Since all the net evaluations are non-negative, the current solution is an optimum one. Hence the optimum solution is : $x_{13} = 5$, $x_{22} = 4$, $x_{26} = 2$, $x_{31} = 1$, $x_{41} = 3$, $x_{44} = 2$, $x_{45} = 4$, $x_{33} = 1$.

The optimum transportation cost is given by

$$z = 5(9) + 4(3) + \Delta\,(7) + 2(5) + 1(6) + 1(9) + 3(6) + 2(2) + 4(2) = \text{Rs. } 112 \quad (\text{since } \Delta \to 0).$$

Note. In above optimum table, Δ may also be introduced in least cost independent cell (2, 5).

Example 17. *Solve the following transportation problem (cell entries represent unit costs) :*

							Available
	5	3	7	3	8	5	3
	5	6	12	5	7	11	4
	2	1	3	4	8	2	2
	9	6	10	5	10	9	8
Required	3	3	6	2	1	2	17 (Total)

[Meerut (M.Com.) Jan. 98 (BP), (M.Sc.) 93 P]

Soluton. Using 'VAM', an initial BFS having the transportation cost Rs. 103 is given below :

Initial BFS

	1(3)				2(5)	3
3(5)			Δ (5)	1(7)		$4 + \Delta = 4$
		2(3)				2
	2(6)	4(10)	2(5)			8
3	3	6	$2 + \Delta = 2$	1	2	

Since the number of allocations (8) in the initial BFS is less than $m + n - 1$ (= 9), introduce negligible quantity Δ in the independent cell (2, 4).

							u_i
	(5) + 2	(3) 1	(7) 0 7	(3) + 2	(8) + 4	(5) 2	−3
	(5) 3	(6) 0 6	(12) + 10	(5) Δ	(7) 1	(11) + 8	0
	(2) + −2	(1) + −1	(3) 2	(4) + −2	(8) + 0	(2) + 1	−7
	(9) + 5	(6) 2	(10) 4	(5) 2	(10) + 7	(9) + 8	0
v_j	5	6	10	5	7	8	

Since all the net-evaluations are non-negative, the current solution is an optimum one. Hence the optimum solution is given by : $x_{12} = 1$, $x_{16} = 2$, $x_{21} = 3$, $x_{25} = 1$, $x_{33} = 2$, $x_{42} = 2$, $x_{43} = 4$ and $x_{44} = 2$.

The optimum transportation cost is

$$z = 1(13) + 2(5) + 3(5) + \Delta\,(5) + 1(7) + 2(3) + 2(6) + 4(10) + 2(5) = \text{Rs. } 103, \text{ as } \Delta \to 0.$$

Example 18. *A company has 4 warehouses and 6 stores; the cost of shipping one unit from warehouse i to store j is* c_{ij}.

If $C = (c_{ij}) = \begin{pmatrix} 7 & 10 & 7 & 4 & 7 & 8 \\ 5 & 1 & 5 & 5 & 3 & 3 \\ 4 & 3 & 7 & 9 & 1 & 9 \\ 4 & 6 & 9 & 0 & 0 & 8 \end{pmatrix}$, *and the requirements of the six stores are 4, 4, 6, 2, 4, 2 and quantities at the warehouse are 5, 6, 2, 9. Find the minimum cost solution.* **[IAS (Main) 95]**

Solution. Using *'lowest cost entry method'*, an initial solution having the transportation cost Rs 70 is obtained as below:

		5(7)				5
	4(1)	Δ (5)			2(3)	6 + Δ = 6
2(4)						2
2(4)		1(9)	2(0)	4(0)		9
4	4	6 + Δ = 6	2	4	2	

Since the number of allocations (8) in the initial BFS is less than $m + n - 1$ (= 9), introduce a negligible quantity Δ in the independent empty cell (2, 3).

Starting Iteration Table

						u_i
+ 7 2	+ (10) 3	5 • (7)	+ (4) −2	+ (7) −2	+ (8) 5	−2
+ (5) 0	4 − θ (1)	Δ + θ (5)	+ (5) −4	(3) −4	2 • (3)	−4
2 − θ (4)	√ + θ −2 (3) 5	−2 (7) 9	+ (9) 0	+ (1) 0	+ (9) 7	0
2 + θ (4)	+ (6) 5	1 − θ (9)	2 • + (0) 5	4 • (0)	+ (8) 7	0
v_j: 4	5	9	0	0	7	

Min. [4 − θ , 2 − θ , 1 − θ] = 0 ⇒ = θ = 1.

Since all the net-evaluations for the non-basic (empty) cells are not non-negative, the initial BFS is not optimal. The empty cell (3, 2) must be allocated the maximum possible amount θ = 1 to this cell. Consequently, cell (4, 3) becomes empty.

First Iteration Table. *Vacate the cell (4, 3) and occupy the cell (3, 2).*

						u_i
+ (7) 4	+ (10) 3	5 • (7)	+ (4) 0	+ (7) 0	+ (8) 5	2
+ (5) 2	3 • (1)	1 • (5)	+ (5) −2	+ (3) −2	2 • (3)	0
1 • (4)	1 • (3)	0 (7) 7	+ (9) 0	+ (1) 0	+ (9) 5	2
3 • (4)	+ (6) 3	+ (9) 7	2 • (0)	4 • (0)	+ (8) 5	2
v_j: 2	1	5	−2	−2	3	

Since all the net evaluations are non-negative, the current solution is optimum. Hence the optimum solution is given by $x_{13} = 5, x_{22} = 3, x_{23} = 1, x_{26} = 2, x_{31} = 1, x_{32} = 1, x_{41} = 3, x_{44} = 2, x_{45} = 4$.

The optimum transportation cost is given by

$z = 5(7) + 3(1) + 1(5) + 2(3) + 1(4) + 3(4) + 1(3) + 2(0) + 4(0) =$ Rs. 68.

Q. 1. Explain "Degeneracy" in a transportation problem. **[Bharathidasan B.Sc (Math) 90; Delhi B.Sc (Math.) 90]**

2. How does the problem of degeneracy arise in a transportation Problem ? Explain how does one overcome it. **[Meerut 94]**

3. Explain how to solve the degeneracy in transportation problems.

EXAMINATION PROBLEMS
(ON DEGENERACY)

1. ***Explain :*** (i) a method of constructing a basic solution of a transportation problem, (ii) the technique of improvement of the constructed basic solution. Solve the following transportation problem :

Cost-matrix

	To			Available
From	0	2	0	70
	1	4	0	30
	0	2	4	50
Required	70	50	30	150

[**Hint.** Use 'VAM' to find initial BFS and prove it to be optimal.]

[**Ans.** $x_{11} = 20$, $x_{12} = 50$, $x_{23} = 30$, $x_{31} = 50$; min. cost = Rs. 100].

2. Determine the optimal solution to each of the following degenerate transportation problem :

(i)

	D_1	D_2	D_3	D_4	D_5	$a_i \downarrow$
O_1	4	7	3	8	2	4
O_2	1	4	7	3	8	7
O_3	7	2	4	7	7	9
O_4	4	8	2	4	7	2
$b_j \rightarrow$	8	3	7	2	2	

[**Ans.** $x_{11} = 1$, $x_{13} = 1$, $x_{21} = 7$, $x_{15} = 2$, $x_{32} = 3$, $x_{33} = 6$, $x_{44} = 2$; min. cost = 56].

(ii) Demand points

		D_1	D_2	D_3	D_4	Supply
	S_1	2	3	11	7	6
Sources	S_2	1	0	6	1	1
	S_3	5	8	15	10	10
Demand		7	5	3	2	17

[**Ans.** $x_{12} = 5$, $x_{13} = 1$, $x_{24} = 1$, $x_{31} = 7$, $x_{33} = 2$, $x_{34} = 1$. min cost = Rs. 102. Alternative solutions also exist.]

(iii)

	To				$a_i \downarrow$
From	10	7	3	6	3
	1	6	7	3	5
	7	4	5	3	7
$b_j \rightarrow$	3	2	6	4	

[**Ans.** $x_{13} = 3$, $x_{21} = 3$, $x_{24} = 2$, $x_{32} = 2$, $x_{33} = 3$, $x_{34} = 2$ min,. cost Rs. 47].

3. A manufacturer has distribution centres located at Agra, Allahabad and Calcutta. These centres have available 40, 20 and 40 units of his product. His retail outlets require the following number of units : *A*, 25; *B*, 10; *C*, 20; *D*, 30; *E*, 15. The shipping cost per unit in rupees between each centre and outlet is given in the following table :

Distribution Centres	Retail Outlets				
	A	B	*C*	*D*	*E*
Agra	55	30	40	50	40
Allahabad	35	30	100	45	60
Calcutta	40	60	95	35	30

Determine the optimal shipping cost.

[**Ans.** $x_{12} = 10$, $x_{13} = 20$, $x_{15} = 10$, $x_{21} = 20$, $x_{31} = 5$, $x_{34} = 30$, $x_{35} = 5$, min. cost = Rs. 3600].

4. A manufacturer wants to ship 8 loads of his product as shown in the table. The matrix gives the mileage from origin *O* to destination *D*. Shipping costs are Rs. 10 per load per mile. What shipping schedule should be used.

	D_1	D_2	D_3	Available
O_1	50	30	220	1
O_2	90	45	170	3
O_3	250	200	50	4
Required	4	2	2	

[**Hint.** Find the initial solution by using Vogel's method. In this solution, number of allocations is less than $m + n - 1$ (*i.e.*, 5). Hence resolve the degeneracy by introducing Δ to one of the empty cells [say, (2, 2)]. Then this initial solution will be optimal with minimum mileage 820 or cost Rs. 8200].

[**Ans.** $x_{11} = 1$, $x_{21} = 3$, $x_{32} = 2$, $x_{33} = 2$]

[IAS (Main) 91]

5. (i) Using *North-West Corner Rule* for initial basic feasible solution, obtain an optimum basic feasible solution to the following degenerate transportation problems :

	To			Available
	7	3	4	2
From	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

[**Ans.** $x_{13} = 2$, $x_{22} = 1$, $x_{23} = 2$, $x_{31} = 4$, $x_{33} = 1$, min. cost = 33]

(iii)

	D_1	D_2	D_3	a_i
O_1	0	2	1	5
O_2	2	1	5	10
O_3	2	4	3	5
$b_j \rightarrow$	5	5	10	

[**Ans.** $x_{21} = 5$, $x_{22} = 5$, $x_{23} = 5$, $x_{33} = 5$ and min. cost = 35]

6. Solve the transportation problem whose cost matrix is given below.

	To				Available
	5	5	4	7	5
	6	4	1	2	5
From	5	9	1	4	6
	8	3	2	4	4
	6	5	3	1	6
Required	5	8	3	10	

[**Ans.** $x_{11} = 2$, $x_{12} = 3$, $x_{22} = 1$, $x_{24} = 4$, $x_{31} = 3$, $x_{33} = 3$, $x_{42} = 4$, $x_{54} = 6$, and min. cost = 73]

7. Obtain the optimal solution by using the best starting solution.

	To				a_i
	10	20	5	7	10
	13	9	12	8	20
From	4	15	7	9	30
	14	7	1	0	40
	3	12	6	19	50
b_j	60	60	20	10	

[**Ans.** $x_{13} = 10$, $x_{22} = 20$, $x_{31} = 30$, $x_{42} = 20$, $x_{43} = 10$, $x_{44} = 10$, $x_{51} = 30$, $x_{52} = 20$ and min cost = 830]

8. A company has 4 warehouses and 6 stores., the surplus in the warehouses, the requirements of the stores and costs (in Rs) of transporting one unit of the commodity from warehouse *i* to the store *j* are given below. How should the commodity be transported so that the total transportation cost is a minimum ? Obtain the initial program by applying the north-west corner rule :

Warehouse	Store						Surplus
	1	2	3	4	5	6	
1	7	5	9	5	10	7	30
2	7	8	24	7	9	13	40
3	4	10	5	6	10	4	20
4	11	8	12	7	12	11	80
Requrement	30	30	60	20	10	20	170

[Delhi B.Sc. (Math.) 90]

[**Ans.** $x_{12} = 10$, $x_{16} = 20$, $x_{21} = 30$, $x_{25} = 10$, $x_{33} = 20$, $x_{42} = 20$, $x_{43} = 40$, $x_{44} = 20$, and min. cost = Rs. 1370]

9. Given below is the unit costs array with supplies a_i, $i = 1, 2, 3$ and demands b_j, $j = 1, 2, 3, 4$,

		Sink				
		1	2	3	4	a_i
	1	8	10	7	6	50
source	2	12	9	4	0	40
	3	9	11	10	8	30
	b_j	25	32	40	23	120

Find the obtimal solution to the above Hitchcock problem. **[Meerut 2002]**

11.12. UNBALANCED TRANSPORTATION PROBLEMS

So far we have discussed the *balanced* type of transportation problems where the total destination requirement equals the total origin capacity (*i.e.*, $\Sigma a_i = \Sigma b_j$). But, sometimes in practical situations, the demand may be more than the availability or *vice versa* (*i.e.* $\Sigma a_i \neq \Sigma b_j$).

Thus, if in a transportation problem, the sum of all available quantities is not equal to the sum of requirements, that is, $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then such problem is called an ***unbalanced transportation problem.***

11·12-1. To Modify Unbalanced T.P. to Balanced Type

An unbalanced T.P. may occur in two different forms. (i) *Excess of availability,* (ii) *Shortage in availability.*

We now discuss these two cases by considering our usual m-origin, n-destination T.P. with the condition that $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$.

Case 1. (Excess Availability, i.e. $\Sigma a_i \geq \Sigma b_j$).

The general T.P. may be stated as follows :

Minimize $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\; c_{ij}$, subject to the constraints

$$\sum_{j=1}^{n} x_{ij} \leq a_i\,, \qquad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j\,, \qquad j = 1, 2, \ldots, n$$

and $\qquad x_{ij} \geq 0\,, \qquad i = 1, 2, \ldots, m;\; j = 1, 2, \ldots, n\,.$

The problem will possess a feasible solution if $\Sigma a_i \geq \Sigma b_j$. In the first constraint, the introduction of slack variable $x_{i,\,n+1}$ $(i = 1, 2, \ldots, m)$ gives

$$\sum_{j=1}^{n} x_{ij} + x_{i,\,n+1} = a_i\;;\; i = 1, 2, \ldots, m$$

or $$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} + x_{i,\,n+1}\right) = \sum_{i=1}^{m} a_i \quad \text{or} \quad \sum_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij}\right) + \sum_{i=1}^{m} x_{i,\,n+1} = \sum_{i=1}^{m} a_i$$

or $$\sum_{i=1}^{m} x_{i,\,n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j = \textit{Excess of Availability.} \left(\because \sum_{i=1}^{m} x_{ij} = b_j\right)$$

If this *excess availability* is denoted by b_{n+1}, the modified general T.P. can be reformulated as :

Minimize $\quad z = \sum_{i=1}^{m} \sum_{j=1}^{n+1} x_{ij}\,(c_{ij})$, subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} + x_{i,\,n+1} = a_i\,, \qquad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j\,, \qquad j = 1, \ldots, n+1$$

$$x_{ij} \geq 0\,, \;\text{ for all } i \text{ and } j\,,$$

where $c_{i,\,n+1} = 0$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n+1} b_j$.

This is clearly the *balanced T.P.* and thus can be easily solved by transportation algorithm.

Working Rule : *Whenever $\Sigma a_i \geq \Sigma b_j$, we introduce a dummy destination-column in the transportation table. The unit transportation costs to this dummy destination are all set equal to zero. The requirement at this dummy destination is assumed to be equal to the difference $\Sigma a_i - \Sigma b_j$.*

Case 2. (Shortage in Availability, i.e. $\Sigma a_i \leq \Sigma b_j$).

In this case, the general T.P. becomes :

Minimize $\quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\,(c_{ij})$ subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \le b_j, \qquad j = 1, \ldots, n$$

$$x_{ij} \ge 0, \qquad i = 1, \ldots, m; j = 1, \ldots n.$$

Now, introducing the slack variable $x_{m+1,j}$ ($j = 1, \ldots, n$) in the second constraint, we get

$$\sum_{i=1}^{m} x_{ij} + x_{m+1,j} = b_j, \qquad j = 1, \ldots, n$$

or
$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij} + x_{m+1,j}\right) = \sum_{j=1}^{n} b_j \quad \text{or} \quad \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) + \sum_{j=1}^{n} x_{m+1,j} = \sum_{j=1}^{n} b_j$$

or
$$\sum_{j=1}^{n} x_{m+1,j} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{n} a_i = \textit{shortage in availability } a_{m+1}, \text{ say.}$$

Thus the modified general T.P. in this case becomes :

$$\text{Minimize} \quad z = \sum_{i=1}^{m+1} \sum_{j=1}^{n} x_{ij} c_{ij},$$

subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i \quad , \quad i = 1, \ldots, m+1$$

$$\sum_{i=1}^{m} x_{ij} + x_{m+1,j} = b_j \quad , \quad j = 1, \ldots, n$$

$$x_{ij} \ge 0 \quad , \quad i = 1, \ldots, m+1; j = 1, \ldots, n.$$

where $c_{m+1,j} = 0$ for $j = 1, \ldots, n$ and $\sum_{i=1}^{m+1} a_i = \sum_{j=1}^{n} b_j$.

Working Rule : *Whenever $\Sigma a_i \le \Sigma b_j$, introduce a dummy source in the transportation table. The cost of transportation from this dummy source to any destination are all set equal to zero. The availability at this dummy source is assumed to be equal to the difference ($\Sigma b_j - \Sigma a_i$).*

Thus, an unbalanced transportation problem can be modified to balanced problem by simply introducing a fictitious *sink* in the first case and a fictitious *source* in the second. The inflow from the source to a fictitious sink represents the surplus at the source. Similarly, the flow from the fictitious source to a sink represents the unfilled demand at that sink. For convenience, costs of transporting a unit item from fictitious sources or to fictitious sinks (as the case may be) are assumed to be zero. The resulting problem then becomes balanced one and can be solved by the same procedure as explained earlier. The method for dealing with such type of problems will be clear in *Example 19.*

Q. 1. What is unbalanced transportation problem ? How do you start in this case ?

2. Explain the technique used to solve the transportation problem with the following restrictions imposed separately in each problem :
   (i) it is required to have all basic solutions non-degenerate,
   (ii) it is required to have no allocation in the (i, j) th cell.
   (iii) it is required to have some positive allocation in the (i, j) th cell.
   (iv) it is found that the total units ready to ship be less than the total units required.

Example 19. *XYZ tobacco company purchases tobacco and stores in warehouses located in the following four cities :*

Warehouse location	Capacity (tonnes)
City A	90
City B	50
City C	80
City D	60

The warehouses supply tobacco to cigarette companies in three cities thay have the following demand :

Cigarette company	Demand (tonnes)
Bharat	120
Janta	100
Red Lamp	110

The adjoining railroad shipping costs per tonne (in hundred rupees) have been determined :

From \ To	Bharat	Janta	Red Lamp
A	7	10	5
B	12	9	4
C	7	3	11
D	9	5	7

Because of railroad construction, shipments are temporarily prohibited from warehouse at city A to Bharat Cigarette Company.

(i) Find the optimum distribution for XYZ Tobacco Company.

(ii) Are there multiple optimum solutions ? If there are alternative optimum solutions, identify them.

(iii) Write the dual of the given transportation problem and use it for checking the optimum solution.

[Pubjab (M.B.A.,) 97)

Solution. (i) The given information is exhibited in the following table. Further, the problem is balanced by adding a dummy row and the cost element (of 7) in the given table corresponding to the cell (*A*—Bharat) is replaced by *M*, since the route is prohibited. Using VAM, the initial feasible solution is obtained and which, when tested for optimality is found to be optimum.

Initial Soln. is Optimal

	Bharat	Janta	Red Lamp	Supply	u_i
A	+ (*M* − 13) (*M*) 13	+ 1 (10) 9	90 (5)	90	1
B	30 (12)	+ 1 (9) 8	20 (4)	50	0
C	0 (7) 7	80 (3) 3	+ 12 (11) − 1	80	− 5
D	40 (9)	20 (5)	+ 6 (7) 1	60	− 3
Dammy	50 (0)	+ 4 (0) − 4	+ 8 (0) − 8	50	− 12
Demand	120	100	110		
$v_j \rightarrow$	12	8	4		

Since opportunity cost in all the unoccupied cells is positive, initial solution shown in the above table is also an optimum solution. The total transport cost associated with this solution is :

$$5 \times 90 + 12 \times 30 + 4 \times 20 + 3 \times 80 + 9 \times 40 + 5 \times 20 = 1{,}59{,}000$$

(b) Since opportunity cost in cell (C, Bharat) $\Delta_{31} = 0$, there exists an alternative optimum solution :

$$x_{13} = 90, x_{21} = 30, x_{23} = 20, x_{31} = 40, x_{42} = 60, \text{total cost} = \text{Rs. } 1{,}59{,}000.$$

(c) The dual of the given problem is :

Maximize $Z = (90\,u_1 + 50\,u_2 + 80\,u_3 + 60\,u_4 + 50\,u_5) + (120\,v_1 + 100\,v_2 + 110\,v_3)$

subject to the constraints

$$
\begin{array}{lll}
u_1 + v_1 \le M, & u_2 + v_3 \le 4 & u_4 + v_2 \le 5 \\
u_1 + v_2 \le 10 & u_3 + v_1 \le 7 & u_4 + v_3 \le 7 \\
u_1 + v_3 \le 5 & u_3 + v_2 \le 3 & \\
u_2 + v_1 \le 12 & u_3 + v_3 \le 11 & \\
u_2 + v_2 \le 9 & u_4 + v_1 \le 9 &
\end{array}
$$

u_i, v_j unrestricted in sign ($i = 1, 2, 3,$ and $j = 1, 2, 3, 4$). Using optimum values of u_i's and v_j's in the objective function, we get

$$90\,(1) + 50\,(0) + 80\,(-5) + 60\,(-3) + 50\,(-12) + 120 \times 12 + 100 \times 8 + 110 \times 4 = \text{Rs. } 1{,}59{,}000,$$

which is the same value as obtained earlier.

Example 20. *A company has four terminals U, V, W and X. At the start of a particular day 10, 4, 6 and 5 trailors are available at these terminals. During the previous night 13, 10, 6 and 6, trailors respectively were loaded at plants A, B, C* and *D. The company despatcher has come up with the costs between the terminals and plants as follows :*

The sale price in rupees per unit and the demand in kg. per unit time are as follows :

Sales centre	Sale price (Rs.) per unit	Demand (kg.) per unit
1	15	120
2	14	140
3	16	60

Find the optimum sales distribution. **[Delhi (MCI) 2000; C.A., Nov. 97]**

Solution : The profit matrix is obtained along with demand and supply and the balanced matrix is given below :

Table 11.73 : Profit Matrix

Factory	Sales Centre 1	2	3	Supply
A	3	2	4	100
B	0	−1	1	20
C	4	3	5	60
D	2	1	3	80
E (Dummy)	0	0	0	60
Demand	120	140	60	320

The table 11.73 representing profit can be converted to an equivalent minimization of loss by subtracting all the profit values in the table from the highest profit value (i.e., 5). The initial basic feasible solution can be found by using Vogel's method. Then obtain optimum solution.

[Ans. $x_{11} = 4$, $x_{13} = 6$, $x_{22} = 3$, $x_{24} = 1$, $x_{32} = 6$, $x_{44} = 5$, **(Dummy)** $x_{51} = 9$, $x_{52} = 1$. **Min. transportation cost is Rs. 555.]**

Example 21. *A multi-plant company has three manufacturing plants, A, B and C and two markets X and Y. Production cost at A, B and C is Rs. 1,500; 1,600 and 1,700 per piece respectively. Selling prices in X and Y are Rs. 4,400 and Rs. 4,700 respectively. Demands in X and Y are 3,500 and 3,600 pieces respectively. Production capacities at A, B and C are 2,000; 3,000 and 4,000 pieces respectively. Transportation costs are as shown in the adjacent table. Build a mathematical model.*

Plant	Market X	Y
A	1,000	1,500
B	2,000	3,000
C	1,500	2,500

Solution. Here three plants differ in production cost and we are given the selling prices also. Therefore, our problem is to determine the schedule of production which may result in the maximum profit. The various profits per item are as shown in the adjacent table.

Profit Matrix

Plant	Market X	Y
A	1,900	1,700
B	800	100
C	1,200	500

The profit (selling price – production cost – transportation cost) data from plants to markets are shown below :

From A to X : 4400 – 1500 – 1000 = 1900; from A to Y : 4700 – 1500 – 1500 = 1700; from B to X : 4400 – 1600 – 2000 = 800; and so on.

Further, total production at A, B and C plants is 2,000 + 3,000 + 4,000 = 9,000 units while total requirement at X and Y is 3,500 + 3,600 = 7,100 units. Hence this is an unbalanced transportation problem. By introducing a dummy market Z to receive an excess production of 9,000 – 7,100 = 1,900 units, the complete relevant information is summarized in the following table :

	Market			Supply
	X	Y	Dummy	
Plant A	(1900)	(1700)	(0)	2000
Plant B	(800)	(100)	(0)	3000
Plant C	(1200)	(500)	(0)	4000
Demand	3500	3600	1900	9000

Let x_{ij} be the quantity to be transported from plant i (i = 1, 2, 3) to market j (j = 1, 2, 3).

Now the mathematical model based on the given data can be formulated as follows :

Maximize (total profit) $\boldsymbol{Z = 1900x_{11} + 1700x_{12} + 800x_{21} + 100x_{22} + 1200x_{31} + 500x_{32}}$

subject to the constraints

$$\left.\begin{aligned} x_{11}+x_{12}+x_{13} &= 2000 \\ x_{21}+x_{22}+x_{23} &= 3000 \\ x_{31}+x_{32}+x_{33} &= 4000 \end{aligned}\right\} \textit{(Supply constraints)} \qquad \left.\begin{aligned} x_{11}+x_{21}+x_{31} &= 3500 \\ x_{12}+x_{22}+x_{32} &= 7600 \\ x_{13}+x_{23}+x_{33} &= 1900 \end{aligned}\right\} \textit{(Demand constraints)}$$

$$\textit{and} \quad x_{ij} \geq 0 \textit{ for } i \textit{ and } j.$$

Example 22. *A steel company has three open hearth furnaces and five rolling mills. Transportation cost (rupees per quintal) for shipping steel from furnaces to rolling mills are shown in the following table* :

Table 11·74

Mills

		M_1	M_2	M_3	M_4	M_5	Capacities (in quintals)
	F_1	4	2	3	2	6	8
Furnaces	F_2	5	4	5	2	1	12
	F_3	6	5	4	7	3	14
Requirement (in quintals)		4	4	6	8	8	

[JNTU (B. Tech.) 2003; Meerut (Maths.)2003, 91]

What is the optimal shipping schedule ?

Solution. Since the total requirements of mills are 30 quintals and the total capacities of all furnaces are 34 quintals, the problem is of unbalanced type. Therefore, the problem can be modified as follows.

Step 1. (***Modifying the given problem to balanced type***).

Since the capacities are *four* quintals more than the total requirements, consider a fictitious mill requiring *four* quintals of steel. Thus the modified (balanced) transportation cost matrix becomes :

Table 11·75

	M_1	M_2	M_3	M_4	M_5	M_f	Capacities
F_1	4	2	3	2	6	0	8
F_2	5	4	5	2	1	0	12
F_3	6	5	4	7	3	0	14
Required	4	4	6	8	8	4	34

Step 2. ***(To find the initial solution).***

Applying the Vogel's method in the usual manner, the initial solution is obtained as given below.

Table 11·76

	M_1	M_2	M_3	M_4	M_5	M_f
F_1		4(2)		4(2)		
F_2				4(2)	8(1)	
F_3	4(6)		6(4)			4(0)

This gives the transportation cost = 4(2) + 4(2)+ 4(2) + 4(2) + 8(1) + 4(6) + 6(4) + 4(0) = Rs. 80.

Step 3. ***(To test the initial solution for optimality).***

Since the total number of allocations is 7 (instead of 6 + 3 – 1 = 8), this is a degenerate basic feasible solution. Therefore, allocate an infinitesimal quantity Δ to empty cell (1, 1). Then, proceeding in the usual manner, following tables for testing the optimality of the solution are obtained.

Table 11·77

u_i and v_j

							u_i
	Δ (4)	• (2)		• (2)			0
				• (2)	• (1)		0
	• (6)		• (4)			• (0)	2
v_j	4	2	2	2	1	–2	

Table 11·78

$(u_i + v_j)$ for empty cells

•	•	2	•	1	–2	0
4	2	2	•	•	–2	0
•	4	•	4	3	•	2
4	2	2	2	1	–2	

Table 11·79

$d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells

•	•	1	•	5	8
1	2	3	•	•	2
•	1	•	3	0	•

Since all $d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells are non-negative, the solution under test is optimal. Further, 0 in the cell (3,5) indicates that alternative solutions will also exist.

Example 23. *A company has three plants at locations A, B and C which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500, and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in Rs.) are given below:*

Determine an optimum distribution for the company in order to minimize the total transportation cost.

		To				
		D	E	F	G	H
From	A	5	8	6	6	3
	B	4	7	7	6	5
	C	8	4	6	6	4

[JNTU 2003, 02, 98; AIMS 2002; Kerala B.Sc (Math.) 90]

Solution. In this problem, the total warehouse requirements (= 2500 units) is greater than the total plant capacity (= 2200 units). Therefore, the problem is of unbalanced type. So introduce a dummy plant P having all transportation costs equal to zero and having the plant availability equal to (2500 – 2200) = 300 units. The modified transportation table is thus obtained as below:

		To					
		D	E	F	G	H	Plant Capacity
From	A	5	8	6	6	3	800
	B	4	7	7	6	5	500
	C	8	4	6	6	4	900
	P	0	0	0	0	0	300
Reuirements		400	400	500	400	800	

Using 'VAM' the following initial BFS is obtained:

	D	E	F	G	H	
A			500(6)		300(3)	800
B	400(4)			100 (6)	Δ (5)	500 + Δ = 500
C		400(4)			500(4)	900
P				300(0)		300
	400	400	500	400	800 + Δ = 800	

Since the number of basic cell allocations (= 7) is less than $m + n - 1$ (= 8), the solution is degenerate. To make the number of allocations equal to 8, introduce a negligible small positive quantity Δ in the independent cell (2, 5). Now test the current solution for optimality.

Starting Table

					u_i
(5) + 2	(8) + 3	(6) $500 - \theta$	(6) + 4	(3) $300 + \theta$	-2
(4) 400	(7) + 5	(7) -1 8	(6) $100 + \theta$	(5) $\Delta - \theta$	0
(8) + 3	(4) 400	(6) -1 7	(6) + 5	(4) 500	-1
(0) + -2	(0) + -1	(0) -2 $+\theta$ √ 2	(0) $300 - \theta$	(0) + -1	-6
$v_j \rightarrow$ 4	5	8	6	5	

Here $\theta = \min[500, \Delta, 300] = \Delta$. So enter the non-basic cell (4, 3) and leave the basic cell (2, 5).

First Iteration Table. Vacate the cell (2, 5) and ocupy the cell (4, 3).

					u_i
(5) + 4	(8) + 3	(6) $500 - \theta$	(6) 0 6	(3) $300 + \theta$	0
(4) 400	(7) + 3	(7) + 6	(6) 100	(5) + 3	0
(8) + 5	(4) 400	(6) -1 7	(6) -1 $+\theta$ √ 7	(4) $500 - \theta$	1
(0) + -2	(0) + -3	(0) $\Delta + \theta$	(0) $300 - \theta$	(0) + -3	-6
v_j 4	3	6	6	3	

Here $\min[500 - \theta, 500 - \theta, 300 - \theta] = 0$ or $300 - \theta = 0$ or $\theta = 300$.

So introduce the cell (3, 4) and drop the cell (4, 4) in the next iteration.

Second Iteration Table. *Vacate the cell (4, 4) and occupy the cell (3, 4).*

					u_i
(5) + 3	(8) + 3	(6) $200 - \theta$	(6) + 5	(3) $600 + \theta$	-1
(4) 400	(7) + 4	(7) 0 7	(6) 100	(5) + 4	0
(8) + 4	(4) 400	(6) -1 $+\theta$ √ 7	(6) 300	(4) $200 - \theta$	0
(0) + -3	(0) + -3	(0) 300	(0) + -1	(0) + -3	-7
$v_j \rightarrow$ 4	4	7	6	4	

Here min $[200-\theta, 200-\theta]=0 \Rightarrow \theta=200$.

So introduce the cell (3, 3) and drop the cell (1, 3) or (3, 5) in the next iteration.

Third Iteration Table. *Vacate the cell (1, 3) or (3, 5) and occupy the cell (3, 3).*

Optimum Table

					u_i
+ (5) 4	+ (8) 4	0 • (6)	0 (6) 6	800 • (3)	0
400 • (4)	+ (7) 4	+ (7) 6	100 • (6)	+ (5) 3	0
+ (8) 4	400 • (4)	200 • (6)	300 • (6)	+ (4) 3	0
+ (0) −2	+ (0) −2	300 • (0)	0 (0) 0	+ (0) −3	−6
v_j: 4	4	6	6	3	

Since all the net evaluations are non-negative, the optimum solution is :

$x_{13}=0, x_{15}=800, x_{21}=400, x_{24}=100, x_{32}=400, x_{33}=200, x_{34}=300, x_{43}=300.$

The optimum transportation cost is given by

$z=0\,(6)+800\,(3)+400\,(4)+100\,(6)+400\,(4)+200\,(6)+300\,(6)+300\,(0)=\text{Rs. } 9200.$

Example 24. *The Bombay Transport Company has trucks available at four different sites in the following numbers :*

Site	:	*A*	*B*	*C*	*D*
No. of Trucks	:	5	10	7	3

Customers W, X, and Y require trucks as shown :

Customer	:	*W*	*X*	*Y*
No. of Trucks	:	5	8	10

Variable costs of getting trucks to the customers are :

From A to W $\rightarrow$ *Rs* 7, *to X* $\rightarrow$ *Rs* 3, *to Y* $\rightarrow$ *Rs* 6; *From B to W* $\rightarrow$ *Rs* 4, *to X* $\rightarrow$ *Rs* 6, *to Y* $\rightarrow$ *Rs.* 8;
From C to W $\rightarrow$ *Rs* 5, *to X* $\rightarrow$ *Rs* 8, *to Y* $\rightarrow$ *Rs.* 4; *From D to W* $\rightarrow$ *Rs* 8, *to X* $\rightarrow$ *Rs* 4, *to Y* $\rightarrow$ *Rs.* 3.

Solve the above transportation problem.

Solution. Since the availability of trucks is greater than the requirement of trucks, the problem is of unbalanced type. Therefore, a dummy requirement column having all the transportation costs equal to zero has been introduced with $25-23\ (=2)$ trucks as its requirement. The following balanced transportation table is then constructed :

		To				
		W	*X*	*Y*	*Z*	Available
	A	7	3	6	0	5
	B	4	6	8	0	10
From	*C*	5	8	4	0	7
	D	8	4	3	0	3
Requirement		5	8	10	2	

Now, using 'VAM', the initial basic feasible solution having transportation cost Rs 58 is obtained :

Initial BFS Table

	5(3)		
5(4)	3(6)	2(8)	
		5(4)	2(0)
		3(3)	

Starting Table. The solution under test is not optimum. The most negative cell evaluation is -4, so introduce the cell (2, 4).

Here, $\min [2-\theta, 2-\theta] = 0 \Rightarrow \theta = 2$.

Therefore, drop either the cell (2, 3) or (3, 4). The resulting solution degenerates.

				u_i
+ (7) 1	5 (3)	+ (6) 5	-1 (0) 1	-3
5 (4)	3 (6)	$2-\theta$ (8)	√ -4 $+\theta$ (0) 4	0
+ (5) 0	+ (8) 2	$5+\theta$ (4)	$2-\theta$ (0)	-4
+ (8) -1	+ (4) 1	3 (3) 1	+ (0) -1	-5
v_j 4	6	8	4	

First Iteration Table. Vacate the cell (3, 4) and occupy the cell (2, 4) with $\theta = 2$.

Since all the net-evaluations are non-negative, the current solution is optimum. The optimum solution is given by $x_{12} = 5, x_{21} = 5, x_{22} = 3, x_{23} = 0, x_{24} = 2, x_{33} = 7, x_{43} = 3$.

The optimum transportation cost is given by $z = 5(3) + 5(4) + 3(6) + 2(0) + 7(4) + 3(3)$ = Rs. 90.

Optimum Table

				$u_i \downarrow$
+ (7) 1	5 (3)	+ (6) 5	+ (0) -3	-3
5 (4)	3 (6)	0 (8)	2 (0)	0
+ (5) 0	+ (8) 2	7 (4)	+ (0) -4	-4
+ (8) -1	+ (4) 1	3 (3)	+ (0) -5	-5
4	6	8	0	

$v_j \rightarrow$

Example 25. *Consider the following unbalanced transportation problem :*

Since there is not enough supply, some of the demands at these destinations may not be satisfied. Suppose there are penalty costs for every unsatisfied demand unit which are given by 5, 3 and 2 for destinations 1, 2 and 3 respectively. Find the optimal solution.

		To 1	2	3	Supply
	1	5	1	7	10
From	2	6	4	6	80
	3	3	2	5	15
Demand		75	20	50	

Solution. In this problem, demand exceeds the supply. The problem is of unbalanced type. Therefore, introduce a '*dummy source*' whose transportation costs are given as 5, 3 and 2 respectively, and supply $145 - 105 = 40$. The modified transportation table is then constructed as follows :

		To 1	2	3	Supply
	1	5	1	7	10
From	2	6	4	6	80
	3	3	2	5	15
	4	5	3	2	40
Demand		75	20	50	

Using 'VAM', an initial basic feasible solution having the transportation cost Rs. 595 is obtained as given below :

Initial BFS Table

	10(1)		10
60(6)	10(4)	10(6)	80
15(3)			15
		40(2)	40
75	20	50	

Optimum Table

				u_i
	+ (5) 3	10 • (1)	+ (7) 3	−3
	60 • (6)	10 • (4)	10 • (6)	0
	15 • (3)	+ (2) 1	+ (5) 3	−3
	+ (5) 2	+ (3) 0	40 • (2)	−4
v_j	6	4	6	

Since all the net-evaluations are non-negative, the optimum solution is given by

$x_{12} = 10, x_{21} = 60, x_{22} = 10, x_{23} = 10, x_{31} = 15$ (allocation in dummy row is not considered)

The optimum transportation cost is given by $z = 0(1) + 60(6) + 10(4) + 10(6) + 15(3) =$ Rs. 515.

Example 26. *A company has received a contract to supply gravel for three new construction projects located in towns A, B and C. Construction engineers have estimated required amount of gravel which will be needed at these construction projects :*

Project location	:	A	B	C
Weekly requirement (truck loads)	:	72	102	41

The company has three gravel pits in towns W, X and Y. The gravel required by the construction projects can be supplied by three pits. The amount of gravel which can be supplied by each pit is as follows :

Plant	:	W	X	Y
Amount available (truck loads)	:	76	82	77

The company has computed the delivery cost from each pit to each project site. These costs (in Rs) are shown in the following table :

Schedule the shipment from each pit to each project in such a manner so as to minimize the total transportation cost within the constraints imposed by pit capacities and project requirements. Find the minimum cost. Is the solution unique ? If it is not, find alternative schedule with the same minimum cost.

Project Location

	A	B	C
W	4	8	8
X	16	24	16
Y	8	16	24

Solution. In this problem, total availability exceeds the total requirement of trucks. So the problem is of unbalanced type. Therefore, introduce a dummy project location D where all the transportation costs are zero and the requirement of the new project location is equal to the difference $(235 - 215) = 20$ units. Then the modified (balanced) transportation table becomes :

	A	B	C	D	Available
W	4	8	8	0	76
X	16	24	16	0	82
Y	8	16	24	0	77
Requirement	72	102	41	20	

Now using 'VAM', the initial BFS having the transportation cost of Rs. 2752 is obtained as below.

Initial BFS

	35(8)	41(8)		76
	62(24)		20(0)	82
72(8)	5(16)			77
72	102	41	20	

Starting Iteration. Since all the net-evaluations are not non-negaive, the initial B.F.S. is not optimum and can be improved further.

Here min $[62-\theta, 41-\theta] = 0 \Rightarrow \theta = 41$. So the cell (1, 3) should leave the basis and the non-basic cell (2, 3) must enter the basis.

				$u_i \downarrow$
+ (4) 0	$35+\theta$ (8)	$41-\theta$ (8)	+ (0) −16	8
0 (16) 16	$62-\theta$ (21)	−8 $+\theta$ √ (16) 24	20 • (0)	24
+ 72 • (8)	+ 5 • (16)	+ (24) 16	+ (0) −8	16
$v_j \rightarrow$ −8	0	0	−24	

First Iteration Table. *Vacate the cell (1, 3) and occupy the cell (2, 3).*

Since all the $d_{ij} \geq 0$, the optimum solution is : $x_{12}=76, x_{22}=21, x_{23}=41, \quad x_{31}=72, x_{32}=5$, ($x_{24}=20$ is dummy so it is neglected). This solution gives the minimum cost of Rs. 2424. Further, since $d_{21}=0$ for the empty cell (2, 1), *the solution obtained above is not unique.* If we again transfer $\theta=21$ units to empty cell (2, 1) and vacate the cell (2, 2), we at once obtain the alternative schedule : $x_{12}=76, x_{21}=21, x_{23}=41$, $x_{31}=51, x_{32}=26$ with the same minimum cost Rs. 2424.

				$u_i \downarrow$
+ (4) 0	76 • (8)	+ (8) 0	+ (0) −16	−16
0 (16) 16	21 • (24)	41 • (16)	20 • (0)	0
72 • (8)	5 • (16)	+ (24) 8	+ 0 −8	−8
v_j 16	24	16	0	

Example 27. *A company produces a small component for all industrial products and distributes it to five wholesalers at a fixed delivered price of Rs.* 2·50 *per unit. Sales forecasts indicate that monthly deliveries will be 3000, 3000, 10000, 5000, 4000 units to wholesalers 1, 2, 3, 4 and 5 respectively. The monthly production capacities are 5000, 10000 and 12500 at plants 1, 2 and 3 respectively. The direct costs of production of each unit are Rs. 1, Rs.* 0·90 *and* Re 0·80 *at plants 1, 2 and 3 respectively. The transportation costs of shipping a unit from a plant to a whole-saler are given below* :

Find how many components of each plant supplies to each wholesaler in order to maximize profit.

[CA (May) 2000]

Solution. Since the total capacity of plants is more than the supply to the wholesalers by a quantity $27{,}500 - 25{,}000 = 2{,}500$ units, so the problem is of unbalanced type. Introduce a dummy wholesaler supplying 2500 units with all the transportation costs from the plants to this destination are assumed to be zero. Also the direct costs of production of each unit are given as Re 1, Re. 0·90 and Rs. 0·80 at plants 1, 2 and 3 respectively. The modified balanced transportation problem is now obtained as follows :

		Wholesaler				
		1	2	3	*4*	*5*
Plant	*1*	·05	·07	·10	·15	·15
	2	·08	·06	·09	·12	·14
	3	·10	·09	·08	·10	·15

Wholesaler

		1	2	3	4	5	6	
Plant	*1*	1·05	1·07	1·10	1·15	1·15	0	5,000
	2	0·98	0·96	0·99	1·02	1·04	0	10,000
	3	0·90	0·89	0·88	0·90	0·95	0	12,500
		3,000	3,000	10,000	5,000	4,000	2,500	

For simplicity in compulation, multiply all the transportation costs in the table by 100, and consider 100 units = 1 unit of items. So, simplified transportation table becomes :

Wholesalers

		1	2	3	4	5	6	Capacity
	1	105	107	110	115	115	0	50
Plant	2	98	96	99	102	104	0	100
	3	90	89	88	90	95	0	125
Supply		30	30	100	50	40	25	

Using 'VAM', the initial BFS having transportation cost of Rs 23,730 is obtained as follows :

25(105)					25(0)	50
5(98)	30(96)	25(99)		40(104)		100
		75(88)	50(90)			125
30	30	100	50	40	25	

Find the net-evaluations in the usual manner as shown in the following starting table.

						$\downarrow u_i$
25 • (105)	+ (107) 103	+ (110) 106	+ (115) 108	+ (115) 111	25 • (0)	7
5 • (98)	30 • (96)	25 • (99)	+ (102) 101	40 • (104)	+ (0) −7	0
+ (90) 87	+ (89) 85	75 • (88)	50 • (90)	+ (95) 93	+ (0) −4	−11
$v_j \rightarrow$ 98	96	99	101	104	7	

Since all the net-evaluations in the empty cells are non-negative, the optimum solution is given by

$x_{11} = 2500, x_{21} = 500, x_{22} = 3000, x_{23} = 2500, x_{25} = 4000, x_{33} = 7500, x_{34} = 5000.$

The optimum transportation cost is given by

$z = 2500\,(1{\cdot}05) + 500\,({\cdot}98) + 3000\,({\cdot}96) + 2500\,({\cdot}99) + 4000\,(1{\cdot}04) + 7500\,({\cdot}88) + 5000\,({\cdot}90) = \text{Rs. } 23730.$

Total 25000 units are supplied to the wholesalers at the fixed rate of Rs. 2·50 per unit,

∴ Total Sale = (25000 × 2.50) = Rs. 62,500.

Total production cost of three plants at the rate of Rs. 1, Rs. 0·90 and Rs. 0·80, respectively, becomes

= Rs. (5000 × 1 + 10,000 × ·90 + 12,000 × ·80) = Rs. 23,600.

Hence the net maximum profit to the manufacturer becomes = Rs. 62,500 − (Rs. 23,730 + Rs. 23,600)

= Rs. 15,170 .

Example 28. *A manufacturer of jeans is interested in developing an advertising compaign that will reach four different age groups. Advertising compaigns can be conducted through T.V., Radio and Magazines. The following table gives the estimated cost in paise per exposure for each age group according to the medium employed. In addition, maximum exposure levels possible in each of the media, namely T.V., Radio and Magazines are 40, 30 and 20 millions respectively. Also the minimum desired exposures within each age group, namely 13–18, 19–25, 26–35, 36 and olders are 30, 25, 15 and 10 millions. The objective is to minimize the cost of attaining the minimum exposure level in each age group.*

Media	Age Groups			
	13–18	19–25	26–35	36 and older
T.V.	12	7	10	10
Radio	10	9	12	10
Magazine	14	12	9	12

(i) Formulate the above as a transportation problem, and find the optimal solution.

(ii) Solve this problem if the policy is to provide at least 4 million exposures through T.V. in the 13–18 age group and at least 8 million exposures through T.V. in the age group 19–25. **[C.A. (May) 91]**

Solution. (i) Formulation as a transportation problem.

Media	Age Groups				Max. exposure available (in millions)
	13–18	19–25	26–35	36 and older	
T.V.	12	7	10	10	40
Radio	10	9	12	10	30
Magazine	14	12	9	12	20
Minimum number of exposures required	30	25	15	10	80/90

Since this transportation problem is of unbalanced type, it can be made balanced by introducing a dummy category before applying *Vogel's Approximation Method.*

Initial Solution Table

Age Groups / Media	13–18	19–25	26–35	36 and older	Dummy	Max. exposure (in million)
T.V.		**25** (7)	**5** (10)	**10** (10)		40
Radio	**30** (10)					30
Magazine			**10**(9)		**10** (0)	20
Min. exposures required (million)	30	25	15	10	10	

The initial solution given by VAM is degenerate since there are only 6 allocations. We put a Δ in the least cost independent cell to proceed for optimality let $u_1 = 0$ and we calculate the remaining u_i and v_j's.

Age Groups

Media	13 – 18	19 – 25	26 – 35	36 and older	Dummy	u_i ↓
T.V.	1; (12) 11	25 •; (7)	5 – θ; (10)	10 •; (10)	√ –1; +θ; (0) 1	0
Media Radio	30 •; (10)	3; (9) 6	3; (12) 9	1; (10) 9	• Δ; (0)	– 1
Magazine	4; (14) 10	6; (12) 6	10 + θ; (9)	3; (12) 9	10 – θ; (0)	– 1
v_j →	11	7	10	10	1	

Min. $[5 - \theta, 10 - \theta] = 0$ gives $5 - \theta = 0$ or $\theta = 5$.

Improved Solution Table

					u_i ↓
2; (12) 10	25 •; (7)	1; (10) 9	10 •; (10)	5 •; (0)	0
30 •; (10)	2; (9) 7	3; (12) 9	0; (10) 10	Δ •; (0)	0
4; (14) 10	5; (12) 7	15 •; (9)	2; (12) 10	5 •; (0)	0
10	7	9	10	0	v_j →

Since all d_{ij}'s are non-negative, the improved solution is optimal.

Through T. V., 25 million people must be reached in the age-group 19–25 and 10 million people in the age group 36 & older.

Through Radio, 30 million people must be reached in the age group 13–18.

Through Magazines, 15 million people must be reached in the age grooup 26–35.

Total minimum cost of attaining the minimum exposure level is Rs. 71 lakhs.

Since $d_{24} = 0$, this solution is not unique. Alternative solutions also exist.

(ii) The required solution is given by

4 • (12)	25 • (7)	(10)	10 • (10)	1 • 0	40
26 • (10)	(9)	(12)	(10)	4 • 0	30
(14)	(12)	15 • (9)	(12)	5 • 0	20
30	25	15	10	10	

Total cost for this allotment is Rs. 71.8 lakhs.

Example 29. *A Company wishes to determine an investment strategy for each of the next four years. Five investment types have been selected, investment capital has been allocated for each of the coming four years, and maximum investment levels have been established for each investment type. An assumption is that amounts invested in any year will remain invested until the end of the planning horizon of four years. The following table summarizes the data for this problem. The values in the body of the table represent net return on investment of one rupee upto the end of the planning horizon. For example, a rupee invested in investment type B at the beginning of year will grow to Rs. 1.90 by the end of the fourth year, yielding a net return of Rs. 0.90.*

Investment made at the beginning of year	Investment Type					Rupees available (in 000's)
	A	B	C	D	E	
	NET RETURN DATA					
1	0·80	0·90	0·60	0·75	1·00	500
2	0·55	0·65	0·40	0·60	0·50	600
3	0·30	0·25	0·30	0·50	0·20	750
4	0·15	0·12	0·25	0·35	0·10	800
Maximum Rupees Investment (in 000's)	750	600	500	800	1000	

The objective in this problem is to determine the amount to be invested at the beginning of each year in an investment type so as to maximize the net rupee return for the four year period.

Solve the above transportation problem and get an optimal solution. Also calculate the net return on investment for the planning horizon of four-year period. **[C.A. (May) 93]**

Solution. We observe that this transportation problem is of unbalanced type and it is a maximization problem. The step-by-step procedure is as follows :

Step 1. We balance the transportation problem by introducing a dummy year with net return 0 assigned to A, B, C, D and E .

Investment Type/Net Return Data

Years \ Type	A	B	C	D	E	Available Rs. (in 000's)
1	0·80	0·90	0·60	0·75	1·00	500
2	0·55	0·65	0·40	0·60	0·50	600
3	0·30	0·25	0·30	0·50	0·20	750
4	0·15	0·12	0·25	0·35	0·10	800
Dummy	0	0	0	0	0	1000
Max. Inv. (in 000's)	750	600	500	800	1000	3650

Step 2. Now convert the above *profit matrix* into a *loss matrix* by subtracting all the elements from the largest element Re 1·00 in the table.

Investment Type

Years \ Type	A	B	C	D	E
1	0·20	0·10	0·40	0·25	0
2	0·45	0·35	0·60	0·40	0·50
3	0·70	0·75	0·70	0·50	0·80
4	0·85	0·88	0·75	0·65	0·90
Dummy	1·00	1·00	1·00	1·00	1·00

Step 3. For convenience, we express the net loss data in above table in paise. Thereafter, we obtain the initial solution by VAM and apply transportation algorithm in the following table. Since all d_{ij}*'s* are not non-negative, we assign θ to most negative d_{ij} cell (5, 2).

					u_i ↓
(20) 20 / 0	(10) Δ − θ	(40) 50 / −10	(25) 45 / −20	(0) 500 + θ	−85
(45) 20 / 25	(35) 600	(60) 45 / 15	(40) 35 / 5	(50) 25 / 25	−60
(70) 0 / 70	(75) −5 / 80	(70) 10 / 60	(50) 750	(80) 10 / 70	−15
(85) 250	(88) −7 / 95	(75) 500	(65) 50	(90) 5 / 85	0
(100) 500	(100) √θ −10 / 110	(100) 10 / 90	(100) 20 / 80	(100) 500 − θ	15
v_j → 85	95	75	65	85	

Min [500 − θ, Δ − θ] = 0 gives θ = Δ.

Step 4. Putting θ = Δ, we get the revised solution and again apply optimality test in the following table.

					u_i ↓
(20) 20 / 0	(10) 10 / 0	(40) 50 / −10	(25) 45 / −20	(0) 500	−85
(45) 10 / 35	~~(35)~~ 600	(60) 35 / 25	(40) 25 / 15	(50) 15 / 35	−50
(70) 0 / 70	(75) 5 / 70	(70) 10 / 60	(50) 750	(80) 10 / 70	−15
(85) 250	(88) 3 / 85	(75) 500	(65) 50	(90) 5 / 85	0
(100) 500	(100) Δ	(100) 10 / 90	(100) 20 / 80	(100) 500	15
v_j → 85	85	75	65	85	

Since all d_{ij} are non-negative, the solution under test is optimal. The optimal allocation is given below.

Year	1	2	3	4	5	6
Investment type	E	B	D	A	C	D
Amount (in 000's)	500	600	750	250	500	50

The net return on investment for the planning horizon of four years is given by

$500 \times 1{\cdot}0 + 600 \times 0{\cdot}65 + 750 \times {\cdot}50 + 250 \times (0{\cdot}15) + 500 \times {\cdot}25 + 50 \times {\cdot}35 =$ **Rs. 1445 thousands.**

Example 30. *A leading firm has three auditors. Each auditor can work upto 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours, project 2 will take 140 hours, the project 3 will take 160 hours. The amount per hour that can be billed for assigning each auditor to each project is given in the table.*

Auditor	Project		
	1 (Rs.)	2 (Rs.)	3 (Rs.)
1	1,200	1,500	1,900
2	1,400	1,300	1,200
3	1,600	1,400	1,500

Formulate this as a transportation problem and find the optimal solution. Also find out the maximum total billings during the next month. **[C.A. (May) 95]**

Solution. Formulation. The given problem can be put in the following tabular form of T.P.

		Project 1	2	3	Available
Auditor	1	1,200	1,500	1,900	160
	2	1,400	1,300	1,200	160
	3	1,600	1,400	1,500	160
Required		130	140	160	

The given problem is of unbalanced type. So we introduce a dummy project to balance it, as follows :

		Project 1	2	3	Dummy	Available
Auditor	1	1,200	1,500	1,900	0	160
	2	1,400	1,300	1,200	0	160
	3	1,600	1,400	1,500	0	160
Required		130	140	160	50	480

Here the problem is to maximize the total billing amount of the auditors. So first we convert this maximization problem into a minimization problem by subtracting all the elements of the above payoff matrix from the highest payoff, *i.e.* Rs. 1900. Thus the minimization T.P. becomes :

		Project 1	2	3	Dummy	Available
Auditor	1	700	400	0	1,900	160
	2	500	600	700	1,900	160
	3	300	500	400	1,900	160
Required		130	140	160	50	480

Now apply *VAM* for finding the initial feasible solution. Since it is a degenerate solution, we introduce a very small quantity Δ in the *least cost independent cell* (3, 3) and then apply optimality test in the usual manner. For convenience, we take figures of payoff's matrix in hundreds of rupees (Rs. 00's).

					$u_i \downarrow$
	8 (7) −1	3 (4) 1	160 • (0)	5 (19) 14	−4
	1 (5) 4	110 • (6)	2 (7) 5	50 • (19)	1
	130 • (3)	30 • (5)	Δ • (4)	1 (19) 18	0
$v_j \rightarrow$	3	5	4	18	

Since all d_{ij}'s for non-basic cells are positive, the initial solution obtained above by *VAM* is optimal.

The optimal allocation of projects to auditors and their billing amount is given below. Here an auditor may involve in more than one project as it is clear from the following :

Auditor	1	2	3	3
Project	3	2	1	2
Billing Amount (Rs.)	160 × Rs. 1900 (Rs. 3,04,000)	110 × Rs. 1300 (Rs. 1,43,000)	130 × Rs. 1600 (Rs. 2,08,000)	30 × Rs. 1400 (Rs. 42000)

Hence, the maximum total billing during the next month will be Rs. 6,97,000.

Example 31. *A particular product is manufactured in factories A, B, C and D; and is sold at centres 1, 2 and 3. The cost (in rupees) of product per unit and capacity (in kg.) per unit time of each plant is given below :*

Factory	Cost (Rs.) per unit	Capacity (kg.) per unit
A	12	100
B	15	20
C	11	60
D	18	80

The sale price in rupees per unit and the demand in kg. per unit time are as follows :

Sales centre	Sale price (Rs.) per unit	Demand (kg.) per unit
1	15	120
2	14	140
3	16	60

Find the optimum sales distribution. **[C.A., Nov. 97]**

Solution. The profit matrix is obtained along with demand and supply and the balanced matrix is given below :

Profit Matrix

Factory	Sales Centre			Supply
	1	2	3	
A	3	2	4	100
B	0	− 1	1	20
C	4	3	5	60
D	2	1	3	80
E (Dummy)	0	0	0	60
Demand	120	140	60	320

The *table* representing profit can be converted to an equivalent minimization of loss by subtracting all the profit values in the table from the highest profit value (*i.e.*, 5). The initial basic feasible solution is found by using Vogel's method.

Since the number of occupied cells are 6 which is one less than the required number $m + n - 1 = 7$, the solution is degenerate and after introducing an allocation of Δ to the least cost independent cell $(A, 3)$, the initial solution is tested for optimality in *table* using MODI method.

Sales Centres

Factory	1	2	3	Supply	$u_i \downarrow$
A	100 • (2)	0 (3) 3	Δ • (1)	100	− 1
B	0 (5) 5	20 • (6)	0 (4) 4	20	2
C	0 (1) 1	0 (2) 2	60 • (0)	60	− 2
D	20 • (3)	60 • (4)	0 (2) 2	80	0
Dummy	+ 1 (5) 4	60 • (5)	0 (2) 2	60	1
Demand	120	140	60	320	
$v_j \rightarrow$	3	4	2		

Since there is no negative opportunity cost in the unoccupied cells in *table*, the solution is optimum. The total maximum profit associated with this solution is as follows :

$$100 \times 3 + 20 \times (-1) + 60 \times 5 + 20 \times 2 \times 60 \times 1 + 60 \times 0 \;=\; \text{Rs. } 680.$$

EXAMINATION PROBLEMS

1. Solve the following unbalanced transportation problem (symbols have their usual meanings).
   [IGNOU 2000]
   [Ans. $x_{12} = 5$, $x_{21} = 8$, $x_{23} = 4$, and min. cost = 23]

	D_1	D_2	D_3	a_i
O_1	4	3	2	10
O_2	2	5	0	13
O_3	3	8	6	12
b_j	8	5	4	

2. Find an optimum basic feasible solution to the following transportation problem (use dummy destinations, if needed) :
   [Ans. $x_{12} = 400$, $x_{24} = 350$, $x_{25} = 50$, $x_{31} = 450$,
   $x_{33} = 200$ $x_{35} = 250$ and min cost = 6,100.]

	Destination					
Warehouse	5	4	8	6	5	600
	4	5	4	3	2	400
	3	6	5	8	4	1000
	450	400	200	250	300	

3. A company has three factories I, II, III and four warehouses 1, 2, 3 and 4. The transportation cost (in Rs.) per unit from each factory to each warehouse, the requirements of each warehouse, and the capacity of each factory are given below :
   Find the minimum cost transportation schedule.

	1	2	3	4	Capacity
I	25	17	25	14	300
II	15	10	18	24	500
III	16	20	8	13	600
Requirement	300	300	500	500	

   [Ans. $x_{14} = 300$, $x_{21} = 200$, $x_{22} = 300$, $x_{33} = 500$, $x_{34} = 100$ and min cost = 15,500.]

4. Consider the following transportation problem with the cost matrix :

	1	2	3	4	5	6	Available
1	5	1	5	8	9	7	30
2	4	3	1	9	2	2	40
3	2	1	3	2	8	2	10
4	1	0	2	8	6	3	110
Required	50	20	10	35	15	50	

   (i) Determine a shipping scheme by any method.
   (ii) Test the above solution for optimality.
   (iii) If the above solution is not optimal, find a better solution (you need not to find the optimal solution)
   [Ans. (i) $x_{13} = 10$, $x_{14} = 10$, $x_{25} = 15$, $x_{26} = 25$, $x_{34} = 10$, $x_{41} = 50$, $x_{42} = 20$, $x_{44} = 15$, $x_{46} = 25$.
   (ii) Not optimal.
   (iii) $x_{14} = 20$, $x_{23} = 10$, $x_{25} = 15$, $x_{26} = 15$, $x_{34} = 10$, $x_{44} = 50$, $x_{42} = 20$, $x_{44} = 5$, $x_{46} = 35$.]

5. Consider the transportation problem with the following cost matrix :

Origins	Destinations				Available
	P	Q	R	S	
A	4	6	8	13	50
B	13	11	10	8	70
C	14	4	10	13	30
D	9	11	13	8	50
Required	25	35	105	20	—

   (i) Determine a shipping scheme by the north-west corner rule.
   (ii) Test the above solution for optimality.
   (iii) If the above solution is not optimal, find a better one.
   [Ans. (i) $x_{11} = 25$, $x_{12} = 25$, $x_{22} = 10$, $x_{23} = 60$, $x_{33} = 30$, $x_{43} = 15$, $x_{44} = 20$.
   (ii) No.
   (iii) $x_{11} = 25$, $x_{12} = 5$, $x_{13} = 20$, $x_{23} = 70$, $x_{32} = 30$, $x_{43} = 15$, $x_{44} = 20$; and min. cost = 1465.]

6. XYZ & Co. has provided the following data seeking your advice on optimum investment strategy :

Investment made at the beginning of year	Net Return Data (in Paise) of Selected Investments				Amount Available (lacs)
	P	Q	R	S	
1	95	80	70	60	70
2	75	65	60	50	40
3	70	45	50	40	90
4	60	40	40	30	30
Max. Investment (lacs)	40	50	60	60	–

The following additional information are also provided :

—P, Q, R and S represent the selected investments.

—The company has decided to have four years investment plan.

—The policy of the company is that amount invested in any year will remain so until the end of fourth year.

—The values (paise) in the table represent net return on investment of one rupee till the end of the planning horizon (for example, a rupee in investment P at the beginning of year will grow to Rs. 1.95 by the end of the fourth year, yielding a return of 95 paise.

—Using the above, determine the optimum investment strategy. **[C.A. (Nov.) 96]**

7. The following table gives unit cost of transporting a certain material from supply points *A*, *B*, *C* to demand points 1, 2, 3, 4 and supply capacities at *A*, *B*, *C* and demands at 1, 2, 3, and 4.

From \ To	1	2	3	4	Supply Capacity
A	5	1	7	10	10
B	6	4	6	5	80
C	3	2	5	8	15
Demand	75	20	50	25	—

Due to previous contractual obligation, a minimum of 5 units of the material must be supplied from A to 4. As there is more demand than supply capacity, demands at some of the demand points cannot be satisfied. There are penalty costs for every unsatisfied demand unit which are given by 5, 3, 2 and 4 for demand points 1, 2, 3 and 4 respectively. Find the optimum schedule of transportation and associated transportatiol cost. Is the optimum schedule obtained unique?

[VTU (BE Mech.) 2002]

8. In the state of Behar, in particular region there are five coal mines which produce the following output at the indicated production cost.

Mine	Output m. tons/day	Production cost units of 100 Rs. per metric ton
1	120	25
2	150	29
3	80	34
4	160	26
5	140	28

Before the coal can be sold to the steel making units, it must be cleaned and graded at one of the coal preparation plants, the capacities and operating cost of these 3 plants are as following :

Plants	Capacities metric tons/day	Operating cost units of 100 Rs. per metric ton
A	300	2
B	.2000	3
C	200	3

All coal is transforted by rail at a cost of Rs. 50 per metric ton kilometer and the distance in kilometers from each mine to the transfortation plants are indicated below :

		Distance Kilometer to mines				
		1	2	3	4	5
Preparation Plant	A	22	44	26	52	24
	B	18	16	24	42	48
	C	44	32	16	16	22

[Hint : (i) Using transportation model determine how the output of each mine should be allocated to the three preparation plants to opmize cost.

(ii) Are alternative approacties possible ? If so, what is the logic of Ist allocation in these alternatives ?
(iii) What is degeneracy and when can it happen ? **[IES (Mech. 1998)]**
[**Hint.** Using Transportation Model
Cost matrix per metric tonne is given as (Rs.)

Preparation plant mine	A	B	C	Supply
1	$(25 \times 100 + 2 \times 100 + 22 \times 50) = 3800$	$(2500 + 300 + 18 \times 50) = 3700$	$2500 + 300 + 44 \times 50 = 5000$	120
2	$(29 \times 100 + 200 + 44 \times 50) = 5300$	$(2900 + 300 + 16 \times 50) = 4000$	$2900 + 300 + 32 \times 50 = 4800$	150
3	$3400 + 200 + 26 \times 50 = 4900$	$(3400 + 300 + 42 \times 50) = 4900$	$3400 + 300 + 16 \times 50 = 4500$	80
4	$2600 + 200 + 52 \times 50 = 5400$	$(2600 + 300 + 42 \times 50) = 5000$	$2600 + 300 + 16 \times 50 = 3700$	160
5	$2800 + 200 + 24 \times 50 = 4200$	$(2800 + 300 + 48 \times 50) = 5500$	$2800 + 300 + 22 \times 50 = 4200$	140
Demand	300	200	200	650 / 700

Here Demand > supply, hence add dummy source with zero cost.
Cost metric in hundred's of rupees with dummy source and using **least cost method**

Source \ Plant	A	B	C	Supply
1	(38)	120 (37)	(50)	120/0
2	70 (53)	80 (40)	(48)	150/70/0
3	80 (49)	(49)	(45)	80/0
4	(54)	(50)	160 (37)	160/0
5	100 (42)	(55)	40 (42)	140/100/0
Dummy	50 (0)	(0)	(0)	50/0
Demand	300/200/50/0	200/80/0	200/40/0	700 / 700

11.13. TIME-MINIMIZING TRANSPORTATION PROBLEMS

In time-minimizing transportation problems, the objective is to minimize the time of transportation rather than the cost of transportation. For example in military, the time of supply is considered more valuable than the cost of transportation and, therefore, it is always prefered to minimize the total time of supply and not the cost. Such problems are often encountered in emergency services like military services, hospital management, fire services, etc.

In fact, the time-minimization transportation problems are similar to the cost-minimization transportation problems, except that the unit transportation cost c_{ij} is replaced by the unit time t_{ij} required to transport the items from ith origin to jth destination.

The *feasible plan* (*initial basic feasible solution*) for this problem can also be obtained by using any of the methods discussed in **sec. 11·8.** Now, if T_j is the largest transportation time associated with the jth feasible plan, then we have to find such plan that gives minimum (T_j).

The *step-by-step* procedure for the solution of such problems is given below.

11·13-1. Solution Procedure for Time-Minimization T.P.

Step 1. First, find an initial basic feasible solution by using any of the methods discussed in **sec. 11·8.** Enter the solution at the centres of the basic cells.

Step 2. Compute T_j for this basic feasible solution and cross out all the non-basic cells for which $t_{ij} \geq T_j$.

Step 3. Now construct a loop for the basic cells corresponding to T_j in such a way that when the values at the centre of the cells are shifted around, the value at this cell tends towards (not-necessarily) zero and no variable becomes zero. If no such closed path is possible, the solution under test is optimal, otherwise go to ***step 2.***

Step 4. Repeat the procedure until an optimum basic feasible solution is attained.

Q. 1. What are the least time transportation problem ?

2. What are the areas, where the time minimization problems are applied ?

3. State the characteristics of a "loop" of a least-time transportaion problems.

4. Give the solution procedure for solving the least-time transportation problems.

11·13-2. Computational Demonstration of Solution Procedure

The computational procedure is best explained by the following example.

Example 32. *If the matrix elements represent the time, solve the following transportation problem* :

		To				
		D_1	D_2	D_3	D_4	Available
	O_1	10	0	20	11	15
From	O_2	1	7	9	20	25
	O_3	12	14	16	18	5
Required		12	8	15	10	45

Solution.

First Iteration :

Step 1. Using *Vogel's Approximation Method,* find an initial basic feasible solution as given in the ***Table 11.80*** The numbers written in the down-left corners of each cell represent the corresponding times.

Step 2. The times for this feasible plan are : $t_{12} = 0$, $t_{14} = 11$, $t_{21} = 1$, $t_{23} = 9$, $t_{33} = 16$, and $t_{34} = 18$.

$\therefore \quad T_1 = \max. [0, 11, 1, 9, 16, 18] = 18.$

Obviously, all the shipments for this plan are to be completed within the time $T_1 = 18$. So we cross-out the cell (1, 3) and (2, 4) because $t_{13} > T_1$ and $t_{34} > T_1$.

Table 11·80

	D_1	D_2	D_3	D_4
O_1		8 (0)		7 (11)
O_2	12 (1)		13 (9)	
O_3			2 (16)	3 (18)

Table 11.81

					u_i
	(10)	8 − (0)	~~(20)~~	7 + (11)	0
	(1)	(7)	13 (9)	~~(20)~~	0
	(12)	+ (14)	2 (16)	3 − (18)	7
v_j	1	0	9	11	

Step 3. Now the closed path (loop) for the cell (3, 4) for which $t_{34} = T_1 = 18$ is shown in *Table 11.81.* It is evident from ***Table 11.81*** that only 3 units can be shifted around.

Second Iteration :

Step 1. The revised feasible plan is shown in ***Table* 11.82**

Table 11.82

(10)	8 (0)	(20)	10 (11)
12 − (1)	(7)	13 + (9)	(20)
+ (12)	3 (14)	2 − (16)	(18)

Step 2. Here $T_2 = \max\{t_{12}, t_{1\backslash 4}, t_{21}, t_{22}, t_{32}, t_{33}\} = \max.\{0, 11, 1, 9, 14, 16\} = 16$

As all the shipments for this feasible solution are completed within time $T_2 = 16$, we cross-out the cell (3, 4) also since $t_{34} > T_2$.

Step 3. The closed loop starts from cell (3, 3) as shown in ***Table 11.82*** above. Clearly, only 2 units can be shifted around.

Third Iteration

Step. 1. The revised feasible plan is shown in ***Table* 11·83**

Table 11.83

(10)	5 (0)	(20)	10 (11)
10 − (1)	+ (7)	15 (9)	(20)
+ 2 (12)	3 − (14)	(16)	(18)

Step 2. Here $T_3 = \max\{t_{12}, t_{14}, t_{21}, t_{23}, t_{31}, t_{32}\} = \max\{0, 11, 1, 9, 12, 14\} = 14$.

As all the shipments for this feasible plan are completed within time $T_3 = 14$, we cross-out the cell (3, 3) also, since $t_{33} > T_3$.

Step 3. Since $t_{32} = T_3 = 14$, The closed loop starts from the cell (3, 2) as shown in ***Table* 11.83**. Clearly, 3 units can be shifted around.

Fourth Iteration :

Step 1. The next revised feasible plan is shown in ***Table* 11.84**

Table 11·84

(10)	5 (0)	(20)	10 (11)
7 (1)	3 (7)	15 (9)	(20)
5 (12)	(14)	(16)	(18)

Step 2. Here $T_4 = \max\{t_{12}, t_{14}, t_{21}, t_{22}, t_{23}, t_{31}\} = \max\{0, 11, 1, 7, 9, 12\} = 12$.

As all the shipments for this feasible plan are completed within time $T_4 = 12$, we cross-out the cell (3, 2) also since $t_{32} > 12$.

Step 3. Now we connot form any closed loop without increasing the present minimum shipping time. Hence the feasible plan at this stage is optimal.

Thus, all the shipments can be made within 12 time units.

EXAMINATION PROBLEM

1. Solve the following transportation problem, the matrix represents the times t_{ij} :

To

		D_1	D_2	D_3	D_4	Available
	O_1	6	7	3	4	5
From	O_2	7	9	1	2	7
	O_3	6	5	16	7	8
	O_4	18	9	10	2	10
Required		10	5	10	5	30

[JNTU (Mech. & Prod.) 2004]

[**Ans.** Total shipment time is 9 units. Details of plan are :
$x_{11} = 2$ with $t_{11} = 6$; $x_{13} = 3$ with $t_{13} = 3$; $x_{23} = 7$ with $t_{23} = 1$; $x_{31} = 8$ with $t_{31} = 6$; and $x_{42} = 5$ with $t_{42} = 9$.]

11.14. TRANSHIPMENT PROBLEMS

Definition. *A transportation problem in which available commodity frequently moves from one source to another source or destination before reaching its actual destination is called a* ***transhipment problem.***

11·14.1. Main Characteristics of Transhipment Problems

Following are the main characteristics of transhipment problems :

1. The number of sources and destinations in the transportation problem are m and n respectively. But in transhipment problems, we have $m + n$ sources and destinations.
2. If S_i denotes the ith source and D_j denotes the jth destination, then commodity can move along the route $S_i \to D_i \to D_j$, $S_i \to S_j \to D_i \to D_j$, $S_i \to D_i \to S_j \to D_j$, or in various other ways. Clearly, transportation cost from S_i to S_i is zero and the transportation costs from S_i to S_j or S_i to D_i do not have to be symmetrical, *i.e.*, in general, $S_i \to S_j \neq S_j \to S_i$.
3. While solving the transhipment problem, we first obtain the optimum solution to the transportation problem, and then proceed in the same manner as in solving the transportation problems.
4. The basic feasible solution contains $2m + 2n - 1$ basic variables. If we omit the variables appearing in the $(m + n)$ diagonal cells, we are left with $m + n - 1$ basic variables.

Q. 1. Explain transhipment problem.
2. What are the main characteristics of a transhipment problem ?
3. Explain the method of solving the transhipment problem.
4. Indicate how a transhipment problem can be solved as a transportation problem. **[Garhwal 97]**
5. What is transhipment problem ? Show how a transhipment problem can be modelled as a transportation problem. **[Delhi (MBA) 95]**
6. Define a Transhipment problem. How does it defer from a transportation problem. How is it solved ?

11·14-2. Computational Demonstration of Solution Procedure

The computational procedure for solving transhipment problems is best explained by the following example.

Example 33. *Consider the following transhipment problem with two sources and two destinations, the costs for shipment in rupees are given below. Determine the shipping schedule :*

	S_1	S_2	D_1	D_2	
S_1	0	1	3	4	5
S_2	1	0	2	4	25
D_1	3	2	0	1	
D_2	4	4	1	0	
			20	10	30

Solution. Step 1. ***(To get modified transportation problem).*** In the transhipment problem, each given source and destination can be considered a source or a destination. If we now take the quantity available at each of the sources D_1 and D_2 to be zero and also at each of the destinations S_1 and S_2 the requirement to be zero, then to have a supply and demand from all the points (sources or destinations) a fictitious supply and demand quantity termed as *'buffer stock'* is assumed and is added to both supply and demand of all the points. Generally, this buffer stock is chosen equal to $\Sigma\, a_i$ or $\Sigma\, b_j$. In our problem, the buffer-stock comes-out to be 30 units.

Modified Table 11.85

	S_1	S_2	D_1	D_2	Available
S_1	0	1	3	4	35
S_2	1	0	2	4	55
D_1	3	2	0	1	30
D_2	4	4	1	0	30
Required	30	30	50	40	

Step 2. ***(To find initial solution of modified problem)***

By adding 30 units of commodity to each point of supply and demand, an initial basic feasible solution is obtained in Table 11.86 by using *Vogel's Approximation method.*

Starting Table 11.86

				a_i
30 (0)	(1)	(3)	5 (4)	35
(1)	30 (0)	20 (2)	5 (4)	55
(3)	(2)	30 (0)	(1)	30
(4)	(4)	(1)	30 (0)	30
b_j: 30	30	50	40	

Step 3. ***(To apply optimality test)***

The variables u_i ($i = 1, 2, 3, 4$) and v_j ($j = 1, 2, 3, 4$) have been determined by using successively the relations $u_i + v_j = c_{ij}$ for all the basic (occupied) cells. These values are then used to compute the net-evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for all the non-basic (empty) cells. Clearly d_{34} (= − 1) is the only negative quantity. Hence an unknown quantity θ is assigned to this cell (3, 4). After identifying the loop, we find that θ = 5 and that the cell (2, 4) leaves the basis (*i.e.*, becomes empty).

Table 11.87

				u_i
30 (0)	1 (1) 0 + 0	1 (3) 0 + 2	5 (4)	0
1 (1) 0 + 0	30 (0)	20 + θ (2)	5 − θ (4)	0
5 (3) −2 + 0	4 (2) −2 + 0	30 − θ (0)	−1 + θ √ (1) −2 + 4	−2
8 (4) −4 + 0	8 (4) −4 + 0	3 (1) −4 + 2	30 (0)	−4
v_j: 0	0	2	4	

Step 4. *Introduce the cell (3, 4) into the basis and drop the cell (2, 4) from the basis. Then, again test the optimality of the revised solution.*

Since all the current net evaluations are non-negative, the current solution is an optimum one. It is shown in Table 12.88. The minimum transportation cost is :

$$z^* = 5 \times 4 + 25 \times 2 + 5 \times 1 = 75.$$

and the optimum transportation route is as shown below.

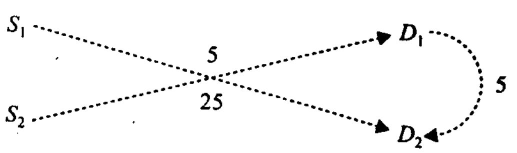

Table 11.88

				u_i
30 • (0)	0 (1) −1	0 (3) 3	5 • (4)	0
2 (1) −1	30 • (0)	25 • (2)	1 (4) 3	−1
6 (3) −3	4 (2) −2	25 • (0)	5 • (1)	−3
8 (4) −4	7 (4) −3	2 (1) −1	30 • (0)	−4
v_j: 0	1	3	4	

EXAMINATION PROBLEMS

1. Given the following data, find the optimum transportation routes :

	S_1	S_2	D_1	D_2	Capacity
S_1	0	2	2	1	8
S_2	1	0	2	3	3
D_1	2	2	0	2	
D_2	1	3	2	0	
Demand			7	4	

[**Ans.** $(S_1, D_1) = 4$ units, $(S_1, D_2) = 4$ units, $(S_2, D_1) = 3$ units; min. cost = 18.]

2. The unit cost of transportation from site i to site j are given below. At site $i = 1, 2, 3$, stocks of 150, 200, 170 units respectively, are available. 300 units are to be sent to site 4 and rest to site 5. Find the cheapest way of doing this :

	1	2	3	4	5
1	–	3	4	13	7
2	1	–	2	16	6
3	7	4	–	12	13
4	8	3	9	–	5
5	2	1	7	5	—

[**Ans.** $x_{14} = 130$, $x_{15} = 20$, $x_{25} = 200$, $x_{34} = 170$; and min. cost = 5070.]

11.15. GENERALIZED TRANSPORTATION PROBLEM

A transportation problem is called generalized transportation problem if the constraints of the general transportation problem are expressed as

$$\sum_{j=1}^{n} a_{ij} x_{ij} = a_i \;\; (1 \le i \le m), \quad \sum_{i=1}^{m} b_{ij} x_{ij} = b_j \;\; (1 \le j \le n) \quad \text{and} \quad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

with $x_{ij} \ge 0$, $(i = 1, 2, ..., m, j = 1, 2, ..., n)$. In the transportation problem, linearly independent equations were $m + n - 1$ in number, and the fact led to the simplicity with which an initial basic feasible solution could be found and tested for optimality. This advantage disappears in the present case. Fortunately, it is still possible to establish an algorithm on the similar lines as that for transportation problem, but is not so simple in practice. Recently, various algorithms have been developed for the solution of generalized problems by many research workers in this field, which are too extensive to be included here.

EXAMINATION QUESTIONS

Q. 1. Indicate how a transhipment problem can be solved as a transportation problem.

2. M sources have s_i, $i = 1, \ldots, M$, units of a commodity while demand at retail shop j is d_j, $j = 1, \ldots, N$. Profit per unit supplied from source i to shop j is p_{ij}, $i = 1, \ldots, M$; $j = 1, \ldots, N$. Indicate how the profit-maximizing problem with the above data can be converted to an equivalent cost-minimizing problem, stating clearly any more assumptions one can make in this regard.

3. What is meant by balanced transportation problem ? Give a method of solving transportation problem with capacities.

4. Explain in detail, any one method for solving a transportation problem. Would you recommend this method to solve an assignment problem.

5. What is degeneracy in transportation problem ? How transportation problems is solved when demand and supply are not equal.

6. State the transportation problem in general terms and explain the problem of degeneracy. How one does evercome it ? **[IAS (Maths.) 90]**

7. What is the total number of constraint equations in a general transportation model with m sources and n destionations ? How many of these are independent ? Justify your answer. Name the methods for obtaining an initial solution. Which is the best one and why ? Also describe it. **[Meerut 2003]**

EXAMINATIONS REVIEW PROBLEMS

1. Formulate the following problem as a Transportation Problem. Do not solve it.

A production manager has a product, which must be produced to meet a fluctuating demand. He knows the monthly requirements are 900 for the Ist month, 700 for the IInd month. 1100 for the IIIrd month, and 1000 for the IVth month. The product can be produced either on regular time or on overtime. However, there are two restrictions imposed by technical conditions : regular production cannot exceed 900 items per month and overtime production cannot exceed 500 items per month. Manufacturing cost per item in normal working time varies each month : Rs. 3 in Ist month, Rs. 4 in IInd month, Rs. 2·50 in IIIrd month, and Rs. 3 in IVth month. The cost varies due to the anticipated volume for each month. The manufacturing costs for overtime are Rs. 4 in Ist month, Rs. 5 in IInd month. Rs. 3·50 in the IIIrd month and Rs. 4 in IVth month. Monthly storage cost is Rs. 2 per unit. Items manufactured but not distributed during the month are kept in stock and distributed the following month. Since the time period is limited, no inventory is to remain at the end of the period (IVth month). The number of units produced must equal to the number demanded and distributed.

2. Solve the following transportation problem :

		Consumers			
		A	B	C	Available
Suppliers	I	6	8	4	14
	II	4	0	8	12
	III	1	2	6	5
	Required	6	10	15	31

[**Hint.** Use 'VAM' to find an initial BFS and prove it to be optimal.]

[**Ans.** $x_{12} = 14$, $x_{21} = 6$, $x_{22} = 5$, $x_{23} = 1$, $x_{32} = 5$, min. cost = Rs. 143].

3. A company has factories at A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are 160, 150 and 190 units respectively. Monthly warehouse requirements are 80, 90, 110 and 160 units respectively. Unit shipping costs (in Rs.) are given in the table. Determine the optimum distribution for this company to minimize shipping costs.

		To			
		D	E	F	G
From	A	42	48	38	37
	B	40	49	52	51
	C	39	38	40	43

[Meerut 2005]

[**Hint.** Unbalanced type problem. Degeneracy may occur. Use 'VAM' for initial BFS and improve it for optimality.]

[**Ans,** $x_{14} = 160$, $x_{21} = 80$, $x_{22} = 10$, $x_{32} = 80$, $x_{33} = 110$, min. cost = Rs. 17050. Alternate solution exists].

4. Given below the unit costs array with supplies a_i, $i = 1, 2, 3$; and demands b_j, $j = 1, 2, 3, 4$. Find the optimum solution to this Hitchcock problem.

		Sink				
		1	2	3	4	a_i
Source	1	8	10	7	6	50
	2	12	9	4	7	40
	3	9	11	10	8	30
	$b_j \rightarrow$	25	32	40	23	120

[**Ans.** $x_{11} = 25$, $x_{12} = 2$, $x_{13} = \Delta$, $x_{25} = 40$, $x_{14} = 23$, $x_{32} = 30$, min. cost = Rs. 840]

5. A company has three factories F_i (i = 1, 2, 3) from which it transports the product to four warehouses W_j (i = 1, 2, 3, 4). The unit cost of production at the three factories are Rs. 4, 3, 5 respectively. Given the following information on unit costs of transportation, capacities at the three factories and requirements at the four warehouses. Find the optimum allocations.

Unit cost of Production		Transportation cost				Available
		W_1	W_2	W_3	W_4	
F_1	4	5	7	3	8	300
F_2	3	4	6	9	5	500
F_3	5	2	6	4	5	200
Required		200	300	400	100	1,000

6. A company has four factories F_1, F_2, F_3 and F_4 and four warehouses, W_1, W_2, W_3 and W_4. The warhouses are located at varying distances from the factories, from where the supplies are transported to them; the transportation costs from the factories to the warehouses thus naturally vary from Rs. 2 to Rs. 6 per unit and the company desires to minimize these transportation costs. In the form of a matrix, the costs from the factories to the warehouses are as shown in the table.

		Warehouses				
		W_1	W_2	W_3	W_4	Capacity
Factories	F_1	2	3	4	5	400
	F_2	3	2	3	4	500
	F_3	4	3	3	4	600
	F_4	6	4	4	5	700
Required		700	600	500	400	

Assign factory capacities to warehouse requirements so as to minimize the costs of transportation by making use of the technique of linear programming.

[Ans. $x_{13} = 300$, $x_{21} = 100$, $x_{22} = 300$, $x_{24} = 100$, $x_{31} = 100$, $x_{33} = 100$, min. cost = Rs. 3200].

7. Origins O_1, O_2, O_3 and O_4 have surplus of 30, 50, 75 and 20 empty freight cars respectively, and the destinations D_1, D_2, D_3, D_4, D_5 and D_6 are in need of 20, 40, 30, 10, 50 and 25 empty cars respectively. The cost per unit of moving an empty car from origins to destinations is given below :

		Destination					
		D_1	D_2	D_3	D_4	D_5	D_6
Origin	O_1	1	2	1	4	5	2
	O_2	2	3	2	1	4	3
	O_3	4	2	5	9	6	2
	O_4	3	1	7	3	4	6

Determine the minimum possible cost of transporting empty cars from 'excess' origins to 'deficit' destinations.

8. XYZ transportation company has four terminals *A*, *B*, *C* and *D*. At the start of a particular day, there are 8, 8, 6 and 3 tractors available at terminals *A*, *B*, *C* and *D*, respectively. During the previous night, trailers were loaded at plants *R*, *S*, *T* and *U* with quantities of 2, 12, 5 and 6 respectively. The company despatcher has come up with the distances (in kms) between the terminals and plants which appear in the table. Based upon the foregoing information, what tractors should the despatcher send to which plants in order to minimize total distances ?

	R	*S*	*T*	*U*
A	22	46	16	40
B	41	15	50	40
C	82	32	48	60
D	40	40	36	30

9. General Electrodes is a big electrode manufacturing company. It has two plants and three main distribution centres in three cities. The supply and demand of transportation for units of electrodes (truck load) are given below along with unit cost of transportation. How the trips be scheduled, so that the cost of transportation is maintained ?

The present cost of transportation is around Rs 3100/– per month. What can be maximum saving by proper scheduling ?

		Centres			
		A	*B*	*C*	Capacity
Plants	*X*	25	35	10	150
	Y	20	5	80	100
Requirement		50	50	150	

10. At the beginning of next week there will be a surplus of 6, 9, 7 and 5 trailers in cities 1, 2, 3 and 4 respectively. Cities *A*, *B* and *C* will have a deficit of 8, 7 and 9 trailers, respectively. The cost for moving from each surplus city to each deficit city is given in rupee in the table.

Deficit

		A	B	C
Surplus	1	26	32	28
	2	19	27	16
	3	39	21	32
	4	18	24	23

Find the Minimum Cost Transportation Schedule.

11. A company has decided to manufacture some or all of five new products at three of their plants. The production capacity of each of these three plants is as follows :

Plant No. :	1	2	3
Production capacity in total number of units :	40	60	90

Sales potential of the five products is as follows :

Product No. :	1	2	3	4	5
Market potential in units :	30	40	70	40	60

Plant No. 3 cannot produce No. 5. The variable cost per unit for the respective plant and product combination is given in the table.

Product

		1	2	3	4	5
Plant	1	20	19	14	21	16
	2	15	20	13	19	16
	3	1	15	18	20	–

Based on the above data, determine the optimum product to plant combination by linear programming.

12. A department store wishes to purchase the following quantities of dresses :

Dress type :	A	B	C	D	E
Quantity :	150	100	75	250	200

Tenders are submitted by four different manufactures who undertake to supply not more than the quantities indicated below :

Manufacturer :	W	X	Y	Z
Total dress quantity :	300	250	150	200

The store estimates that its profit (in Rs.) per dress will vary with the manufacturer as shown in the following table :

	A	B	C	D	E
W	2·75	3·50	4·25	2·25	1·50
X	3·00	3·25	4·50	1·75	1·00
Y	2·50	3·50	4·75	2·00	1·25
Z	3·25	2·75	4·00	2·50	1·75

13. The following information is available concerning the operation of the XYZ manufacturing company :

Period	Units on order	Production Capacity		Excess cost per unit over time (Rs.)	Storage cost per unit (Rs.)
		Regular time	Over time		
Month 1	800	920	920	1·25	0·50
Month 2	1400	250	250	1·25	0·50

Formulate the above problem as a transportation problem.

[Ans.

		1	2	3′	a_i
Regular time	1	1	1·50	0	920
(Over time)	1′	2·25	2·75	0	920
Regular time	2	∞	1	0	250
(Over time)	2′	∞	2·25	0	250
	b_j	800	1400	140	

]

14. A company has factories A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are 250, 300 and 400 units respectively for regular production. If overtime production is utilized, factories A and B can produce 50 and 75 additional units respectively at overtime incremental costs of Rs. 4 and Rs. 5 respectively. The current warehouse requirement are 200, 225, 275 and 300 units respectively. Unit transportation costs (in Rs.) from factories to the warehouses are as below.

To

		D	E	F	G
	A	11	13	17	14
From	B	16	18	14	10
	C	21	24	13	10

Determine the optimum distribution for this company to minimize costs.

[Ans.

	D	E	F	G	H	a_i
A	11	13	17	14	0	250
B	16	28	14	10	0	300
C	21	24	13	10	0	400
A′	15	17	21	18	0	50
B′	21	23	19	15	0	75
b_j	200	225	275	300	75	

]

15. A manufacturer must produce a certain product in sufficient quantity to meet contracted sales in the next four months. The production facilities available for this product are limited, but by different amounts in respective months. The unit cost of production also varies accordingly to the facilities and personnel available. The product may be produced in one month and then held for sale in a later month, but an estimated storage cost of Re. 1 per unit per month. No storage cost is incurred for goods sold in the same month in which they are produced. There is presently no inventory of this product and none is desired at the end of 4 months. Given the following table show how much to produce in each of four months in order to minimize total cost.

Month	Contracted sales (in units)	Maximum production (in units)	Unit cost of production (Rs.)	Unit storage cost per month (Rs.)
1	20	40	14	1
2	30	50	16	1
3	50	30	15	1
4	40	50	17	1

Formulate the problem as a transportation problem and hence solve it.

[Ans.

	1	2	3	4	5	a_i ↓
1	14	15	16	17	0	40
2	0	16	17	18	0	50
3	0	0	15	16	0	30
4	0	0	0	17	0	50
b_j →	20	30	50	40	30	

]

16. Company has four warehouses a, b, c, d. It is required to deliver a product from these warehouses to three customers A, B, and C. The warehouses have the following amounts in stock :

Warehouse	:	a	b	c	d
No. of units	:	15	16	12	13

and the customer's requirements are :

Customer	:	A	B	C
No. of units	:	18	20	18

The table below shows the costs of transporting one unit from warehouse to the customer.

	a	b	c	d
A	8	9	6	3
B	6	11	5	10
C	3	8	7	9

Solve the problem.

[Hint. Use 'VAM' to find initial BFS and prove it to be optimal.]

[Ans. $x_{12} = 5$, $x_{14} = 13$, $x_{22} = 8$, $x_{23} = 12$, $x_{31} = 15$, $x_{32} = 3$, min. cost = Rs. 331]

17. Solve the following transportation problem.

21	16	25	13	11
17	18	14	23	13
32	27	18	41	19
6	10	11	15	43

18. Goods have to be transported from factories F_1, F_2, F_3, to warehouses W_1, W_2, W_3, and W_4. The transportation costs per unit, capacities of the factories and requirements of the warehouses are given in the following table. Find the distribution with minimum cost.

	W_1	W_2	W_3	W_4	Capacity
F_1	15	24	11	12	5000
F_2	25	20	14	16	4000
F_3	12	16	22	13	4000
Requirement	3000	2500	3500	4000	

[Ans. $x_{13} = 2{,}500$, $x_{14} = 2{,}500$, $x_{23} = 1000$, $x_{21} = 3000$, $x_{32} = 2500$, $x_{34} = 1500$ and min. $z = 1{,}67{,}000$.]

19. What is an assignment problem ? Show that an assignment problem can be reduced to transportation problem and vice-versa.

20. What is degeneracy in Transportation Problems ? How transportation problem is solved when demand and supply are not equal ?

21. A company has four Factories from which it ships its product units to four warehouses W_1, W_2, W_3 and W_4 which are the distribution centres. Transportation costs per unit between various combinations of factories (F_1, F_2, F_3 and F_4) and warehouses are as:

	W_1	W_2	W_3	W_4	Available
F_1	48	60	56	58	140
F_2	45	55	53	60	260
F_3	50	65	60	62	360
F_4	52	64	55	61	220
Required	200	320	250	210	

Find the transportation schedule which minimizes the distribution cost.

[Hint. Find the initial solution by VAM as $(F_1, W_2) = 60$, $(F_1, W_3) = 30$, $(F_1, W_4) = 50$ $(F_2, W_2) = 260$ $(F_3, W_1) = 200$ $(F_3, W_4) = 160$, $(F_4, W_3) = 220$ units.

and prove it to be optimum. Since d_{33} will be zero, there will exist alternative solutions also.]

22. Solve the following transportation problem using *north-west corner rule* for initial feasible solution.

A company has 3 Plants P_1, P_2, P_3 each producing 50, 100 and 150 units of a similar product. There are five warehouses W_1, W_2, W_3, W_4 and W_5 having demand of 100, 70, 50, 40 and 40 units respectively.

The cost of sending a unit from various plants to the warehouses differs as given by the cost matrix below. Determine a transportation schedule so that cost is minimized.

	W_1	W_2	W_3	W_4	W_5	a_i
P_1	20	28	32	55	70	50
P_2	48	36	40	44	25	100
P_3	35	55	22	45	48	150
b_j	100	70	50	40	40	300/300

[Ans. $(P_1, W_1) = 40$, $(P_1, W_2) = 10$, $(P_2, W_2) = 60$, $(P_2, W_5) = 40$
$(P_3, W_1) = 60$, $(P_3, W_3) = 50$, $(P_3, W_4) = 40$, min. cost = Rs. 9240].

23. A transportation problem with 3 sources and 4 destinations has :

$C_{11} = 2$, $C_{12} = 3$, $C_{13} = 11$, $C_{14} = 7$, $C_{21} = 1$, $C_{22} = 0$, $C_{23} = 6$, $C_{24} = 1$, $C_{31} = 5$, $C_{32} = 8$, $C_{33} = 15$, $C_{34} = 9$
$S_1 = 6$, $S_2 = 1$, $S_3 = 10$, $D_1 = 7$, $D_2 = 5$, $D_3 = 3$, $D_4 = 2$

Solve this T.P. **[Meerut (Math.) 96]**

24. (a) Give the difference between transportation and assignment problems.

(b) Explain in brief with examples : (i) North west corner rule (ii) Vogel's approximation method.

25. State the Transportation problem in general terms and explain the problem of degeneracy. How does one overcome it? **[IAS (Main) 90]**

26. Four intravenous fluid manufacturing plants are located at S_1, S_2, S_3 and S_4 which supply the Hospitals located at H_1, H_2, H_3, H_4 and H_5. Daily plant capacities are 80, 120, 125 and 75 units respectively. Daily hospital requirements are 30, 70, 150, 50 and 100 units, respectively. Unit transportation costs (in Rs.) are given below:

		To				
		H_1	H_2	H_3	H_4	H_5
	S_1	6	8	15	17	9
From	S_2	11	13	7	4	16
	S_3	13	15	8	6	11
	S_4	10	5	9	3	6

Determine the optimum distribution for the plants in order to minimize the total transportation cost. **[Delhi (MBA) 94]**

27. A company has three warehouses W_1, W_2 and W_3. It is required to deliver a product from these warehouses to three customers A, B and C. The warehouses have the following units in stock.

Warehouse	W_1	W_2	W_3
No. of units	65	42	43

and customer requirements are:

Customer	A	B	C
No. of units	70	30	50

The table below shows the costs of transporting one unit from warehouse to the customer:

		Warehouse		
		W_1	W_2	W_3
Customer	A	5	7	8
	B	4	4	6
	C	6	7	7

Find the optimum transportation route. **(C.A., May 99)**

28. (a) A company is spending Rs. 1,000 on transportation of its units from three plants to four distribution centres. The supply and demand of units with unit cost of transportation are given as:

		Distribution centre				
		D_1	D_2	D_3	D_4	Capacity (tonnes)
	P_1	19	30	50	12	7
Plant	P_2	70	30	40	60	10
	P_3	40	10	60	20	18
	Demand	5	8	7	15	35

What can be the maximum saving by optimum scheduling? **(AIMA, (P.G. Dip. in Management) June 96)**

(b) A company has three cement factories A, B and C, and four area distributors W, X, Y and Z. With identical costs of production at the three factories, the only variable cost involved is the transportation cost. The monthly production capacity (in tonne) of the three factories, monthly demand of the four distributors, and the transportation cost per tonne (in rupees) from the different factories to different distribution centres, are given below:

Factory	Distributor				Monthly availability
	W	X	Y	Z	
A	20	25	50	10	45,000
B	45	50	15	40	50,000
C	22	10	45	35	55,000
Monthly demand	50,000	40,000	30,000	30,000	

Suggest an optimal transport schedule and find the minimum transportation cost. **(Delhi (M.Com.) 99)**

[**Hint:** Take dummy refinary with zero cost and capacity 100 to make the problem balanced. Initial solution by VAM is optimal.
Ans. $x_{11} = 300$, $x_{12} = 200$, $x_{14} = 200$, $x_{22} = 400$, $x_{33} = 700$, $x_{34} = 100$, $x_{44} = 100$, Min. cost = Rs. 700.]

29. A state has four government hospitals A, B, C and D. Their monthly requirements of medicines, etc. are met by four distribution centres X, Y, Z and W. The data in respect of a particular item *via-a-vis* availabilities at the centres, requirements at the hospitals and distribution cost per unit (in paise) is given in the following table :

Distribution centre	Hospital				Availability
	A	*B*	*C*	*D*	
X	44	84	84	80	2,000
Y	92	30	64	80	12,000
Z	32	100	96	72	5,000
W	80	36	120	60	6,000
Requirement	8,000	8,000	6,000	3,000	25,000

Determine the optimum distribution. **(Delhi (MBA) Nov. 98)**

[**Hint :** First find initial solution by VAM.

Ans. $x_{11} = 100$, $x_{23} = 200$, $x_{33} = 450$, $x_{41} = 400$, $x_{53} = 200$, $x_{62} = 350$, $x_{71} = 100$, $x_{72} = 50$, $x_{73} = 150$.]

30. A company must ship from 3 factories to 7 warehouses. The transportation cost per unit from each factory to each warehouse, the requirement of each warehouse, and the capacity of each factory are :

Warehouse	Factory			Warehouse requirement
	1	2	3	
A	6	11	8	100
B	7	3	5	200
C	5	4	3	450
D	4	5	6	400
E	8	4	5	200
F	6	3	8	350
G	5	2	4	300
Factory capacity	600	400	1000	2000

(i) Find the minimum cost schedule.

(ii) Suppose warehouse B goes out of business. This means that there is now excess capacity of 200 units. Find the minimum cost transportation schedule now. **(Kurukshetra (M.B.A) 96)**

31. A firm manufacturing a single product has three plants I, II and III. They have produced 60, 35 and 40 units respectively during this month. The firm had made a commitment to sell 22 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D and 30 units to customer E. Find the minimum possible transportation cost of shifting the manufactured product to the five customers. The net unit cost (in Rs.) of transporting from the three plants to the five customers is given below :

		Customers			
		A	*B*	*C*	*D*
Plants	I	4	1	3	4
	II	2	3	2	3
	III	3	5	2	2

(Delhi (M.B.A.) March 99)

32. Four gasoline dealers A, B, C and D require 50, 40, 60 and 40 KL of gasoline respectively. It is possible to supply these from locations 1, 2 and 3 which have 80, 100 and 50 KL respectively. The cost (in Rs.) for shipping every KL is shown in the table below :

		A	*B*	*C*	*D*
Location	1	7	6	6	6
	2	5	7	6	7
	3	8	5	8	6

Determine the most economical supply pattern. **(Jammu (M.B.A.) 96)**

[**Hint :** Add a dummy dealer to make the problem balanced.

Ans. $x_{13} = 10$, $x_{14} = 30$, $x_{21} = 50$, $x_{23} = 50$, $x_{32} = 40$, $x_{34} = 10$ and min. cost = Rs. 1,050.]

33. Priyanshu Enterprise has three factories at locations *A*, *B* and *C* which supplies three warehouse located at *D*, *E* and *F*. Monthly factory capacities are 10, 80 and 15 units respectively. Monthly warehouse requirements are 75, 20 and 50 units respectively. Unit shipping costs (in Rs.) are given in the following table :

		Warehouse		
		D	*E*	*F*
	A	5	1	7
Factory	*B*	6	4	6
	C	3	2	5

The penalty costs for not satisfying demand at the warehouses *D*, *E* and *F* are Rs. five, Rs. three, and Rs. two per unit respectively. Determine the optimum distribution for Priyanshu, using any of the known algorithms write the dual of this problem. **(Delhi (M.B.A.) March 99)**

[**Hint** : Demand (145) > Supply (105), Add dummy source with supply 40 and transportation costs 5, 3 and 2 for destination 1, 2 and 3 respectively.

Ans. $x_{12} = 0$, $x_{21} = 60$, $x_{22} = 10$, $x_{23} = 10$, $x_{31} = 15$ and $x_{43} = 40$, min. cost = Rs. 514. Penalty for transporting 40 units to destination 3 at the cost of Rs. 2 per unit is Rs. 80.]

34. The Purchase Manager, Mr. Shah, of the State Road Transport Corporation must decide on the amounts of fuel to buy from three possible vendors. The corporation refuels its buses regularly at the four depots within the area of its operations.

The three oil companies have said that they can furnish up to the following amounts of fuel during the coming month : 275,000 litres by oil company 1 : 50,000 litres by oil company 2; and 660,000 litres by oil company 3. The required amount of the fuel is 110,000 litres by depot 1; 20,000 litres at depot 2; 330,000 litres at depto 3; and 440,000 litres at depot 4.

When the transportation costs are added to the bid price per litre supplied, the combined cost per litre for fuel from each vendor servicing a specific depot is shown below :

	Company 1	Company 2	Company 3
Depot 1	5.00	4.75	4.25
Depot 2	5.00	5.50	6.75
Depot 3	4.50	6.00	5.00
Depot 4	5.50	6.00	4.50

Determine the optimum schedule. **[Delhi (MBA) Nov. 97]**

[**Hint** : Supply (1485000) > Demand (1100000), add dummy column with demand of 385000 eithers of oil.

Ans. $x_{13} = 110$, $x_{21} = 55$, $x_{22} = 165$, $x_{31} = 220$, $x_{33} = 110$, $x_{43} = 440$, $x_{52} = 385$, min. transportation cost = Rs. 51,700. Thus required cost = Rs. 515 + 80 + Rs. 595.]

35. (a) A company has three warehouses *A*, *B* and *C* and four stores *W*, *X*, *Y* and *Z*, the warehouses have altogether a surplus of 150 units of a given commodity as followings :

A	50
B	60
C	40

The four stores need the following amounts :

W	20
X	70
Y	50
Z	10

Cost (in rupees) of shipping one unit of commodity from various warehouses to different stores is as follows :

		Store			
		W	*X*	*Y*	*Z*
	A	50	150	70	60
Warehouse	*B*	80	70	90	10
	C	15	87	79	81

Find the optimum distribution schedule and the associated transportation cost. **(Madras (MBA,) Nov. 97)**

(b) A distribution system for a particular kind of medicine has the following characteristics :

Warehouse	Availability (in cases)	Dispensary	Requirement (in cases)
A	55	I	35
B	25	II	65
C	50	III	30

The transportation costs per case (in rupees) associated with each route are :

		To		
		I	II	III
From	*A*	15	12	9
	B	7	8	12
	C	10	7	8

Find the optimum distribution schedule. **(Delhi (MBA) April 98)**

36. An oil company has three refineries that produce total which is then transported to four distribution centres. The total quantity produced by each refinery and the total requirement of each distribution centre and the associated transportation cost per 1000 barrels are given below :

Refinery	Distribution centre				Supply (barrels)
	W	*X*	*Y*	*Z*	
A	80	70	50	60	16,000
B	60	90	40	80	20,000
C	50	50	95	90	14,000
Demand (barrels)	10,00	10,000	12,000	18,000	50,000

(i) Suggest the trasnportation schedule that minimizes the total transportation cost.

(ii) If the company wants that at least 5,000 barrels of oil are transported from Refinery *C* to distribution Centre *W*, will the optimum transportation schedule change ? If so, what will be the new schedule ?

(iii) If the transportation cost from *A* to *W* increases by Rs. 10/- will the solution in (i) change ? Why ?

(Delhi (M.Com.,) 94)

37. A cement company has three factories which manufacture cement which is then transported to four distribution centres. The quantity of monthly production of each factory, the demand of each distribution centre and the associated transportation cost per quintal are given below :

Factory	Distribution centres				Monthly production (in quintals)
	W	*X*	*Y*	*Z*	
A	10	8	5	4	7,000
B	7	9	15	8	8,000
C	6	10	14	8	10,000
Monthly demand (in quintals)	6,000	6,000	8,000	5,000	25,000

(i) Suggest the optimum transportation schedule.

(ii) Is there any other transportation schedule which is equally attractive ? If so, write that.

(iii) If the company wants that at least 5,000 quintals of cement are transported from factory *C* to distribution centre *Y*, will the transportation schedule be any different ? If so, what will be the new optimum schedule and the effect on cost ?

(Delhi (M. Com.) 97)

38. A company has three plants of cement which has to be transported to four distribution centres. With identical costs of production at the three plants, the only variable costs involved are transportation costs. The monthly demand at the four distribution centres and the distance from the plants to the distribution centres (in kms) are given below :

Plant	Distribution centre				Monthly production (tonnes)
	W	X	Y	Z	
A	500	1,000	150	800	10,000
B	200	700	500	100	12,000
C	600	400	100	900	8,000
Monthly demand (tonnes)	9,000	9,000	10,000	4,000	

The transport charges are Rs. 10 per tonne per kilometer. Suggest optimum transportation schedule indicate the total minimum transportation cost. If, for certain reasons, route from plant *C* to distribution centre *X* is closed down, will the transportation scheme change ? If so, suggest the new schedule and effect on total cost. **(Delhi (M. Com.) 98)**

39. A company has three factories *A*, *B* and *C*; and four distribution centres *X*, *X*, *Y* and *Z*. With identical costs of production at the three factories, the only variable costs involved are transportation costs. The monthly production at the three demand at the four distribution centres is 6,000 tonnes, 4,000 tonnes, 2,000 tonnes and 15,000 tonnes, respectively. The transportation cost per tonne from different factories to different centres are given below :

Factory	Distribution Centre			
	W	X	Y	Z
A	3	2	7	6
B	7	5	2	3
C	2	5	4	5

(i) Suggest the optimum transportation schedule. What will be the minimum transporation cost ?

(ii) If the transportation cost from factory *C* to centre *Y* increases by Rs. 2 per tonne, will the optimum transportation schedule change ? Why ?

(iii) If the company wants to transport at least 1,000 tonnes of the product from factory *A* to centre *Z*, will the solution in (i) above change ? If so, what will be the new schedule and the effect on total transportation cost ?

[Delhi (M. Com.), 96]

40. (a) Solve the following transportation problem to maximize profit and give criteria for optimality :

Profit (Rs.)/Unit
Destination

Origin	1	2	3	4	Supply
A	42	27	24	35	200
B	46	37	32	32	60
C	40	40	30	32	140
Demand	80	40	120	60	

[Sardar Patel (MBA) 98]

[**Hint :** Add dummy destination with zero cost and demand of 100 units.
Ans. Max. profit = (40) (42) + (60) (35) + (35) (40) + 46 + (20) (32) + (40) (40) + (100) (30) = Rs. 10,860.]

(b) Solve the following transportation problem to maximize profit and give criteria for optimality. Use Vogel's approximation method for maximization.

Profit (Rs./unit)
Destination

Origin	1	2	3	4	Supply
A	40	25	22	33	100
B	44	35	30	30	30
C	38	38	28	30	70
Demand	40	20	60	30	200 150

41. A departmental store wishes to purchase the following types of sarees :

Types of sarees :	*A*	*B*	*C*	*D*	*E*
Quantity :	150	100	75	250	200

Tenders are submitted by four different manufacturers who undertake to supply not more than the quantities mentioned below (all types of sarees combined) :

Manufacturer :	*W*	*X*	*Y*	*Z*
Total quantity :	300	250	150	200

The store estimates that its profit per saree will vary with the manufacturer as shown below :

Manufacturer	Saree				
	A	*B*	*C*	*D*	*E*
W	275	350	425	225	150
X	300	325	450	175	100
Y	250	350	475	200	125
Z	325	275	400	250	175

Use transportation technique to determine how the orders should be placed ? What is the maximum profit ?
(Punjabi (MBA) 96)

[**Hint :** First add dummy column with demand 125 sarees and subtract all elements of the profit matrix from the highest element 475.
Ans. $x_{12} = 25$, $x_{14} = 50$, $x_{15} = 200$, $x_{16} = 25$, $x_{21} = 150$, $x_{27} = 100$, $x_{32} = 75$, $x_{33} = 75$, $x_{44} = 200$.]

42. A company produces three kinds of dolls *A*, *B* and *C*. The monthly production is 1000 units; 2,000 units and 3,000 units respectively. Those dolls are sold through three departmental stores *X*, *Y* and *Z*. 1,500 dolls are to be supplied to each store every month. However, store *Z* does not want any doll of type *A*. Profit per unit of dolls sold to each of the stores is given below :

		Stores		
		X	*Y*	*Z*
Dolls	*A*	5	10	—
	B	16	8	9
	C	12	9	11

Suggest optimum policy schedule and the profit element. **(Kerala (M.Com.) 97)**

43. The City Super Market keeps five different patterns of a particular size of readymade garments for sales. There are four manufacturers available to the Manager of the market to order the required quantity. The demand for the coming season and the maximum quantity that can be produced by a manufacturer is as follows :

Pattern	1	2	3	4	5
	U. Cut	P. Form	Maxi	Mini	Midi
Quantity required	200	150	200	150	300
Manufacturer	*A*	*B*	*C*	*D*	
Capacity	200	300	250	250	

The following quotations have been submitted by the manufacturers for different patterns :

Manufacturer	1	2	3	4	5
A	Rs. 110	Rs. 130	Rs. 160	Rs. 70	Rs. 140
B	100	120	150	75	130
C	115	140	155	80	120
D	125	145	140	60	125

The Super Market is selling different patterns at the following prices :

Pattern :	1	2	3	4	5
Selling Price (Rs.) :	120	150	170	80	140

How should the Super Market Manager place the order ?

44. A leading firm has three auditors. Each auditor can work up to 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours, project 2 will take 140 hours, and project 3 will take 160 hours. The account per hour that can be billed for assigning each auditor to each project is given below :

Auditor	Project 1	2	3
	Rs.	Rs.	Rs.
1	1,200	1,500	1,900
2	1,400	1,300	1,200
3	1,600	1,400	1,500

FORMULATE this as a transportation problem and find the optimum solution. Also find out the maximum total billings during the next month. **(C.A., May 95)**

[**Hint :** First convert the problem from maximization to minimization by subtracting all elements from the highest pay off Rs. 1900.

Ans. Here an auditor may be involved in more than one project as is apparent from the following :

Auditor	Project	Rate	Hours	Billings (Rs.)
1	3	1900	160	304000
2	2	1300	110	143000
3	1	1600	130	208000
3	2	1400	30	42000
				697000

45. A company has four factories situated in four different locations in the country and four sales agencies located in four other locations in the country. The cost of production (Rs. per unit), the sale price, capacities and monthly requirement are given below :

Factory	Sales Agency				Monthly capacity (units)	Cost of production
A	7	5	6	4	10	10
B	3	5	4	2	15	15
C	4	6	4	5	20	16
D	8	7	6	5	15	15
Monthly requirement (units)	8	12	18	22		
Sales price	20	22	25	18		

Find the monthly production and distribution schedule which will maximize profit. **(C.A., May 96)**

[**Hint :** Profit = Sales price – Production cost – Shipping cost.

Ans. The allocation of factories to sales agencies and their profit amount is given below :

From	To	Quantity	Profit	Total Profit
A	2	10	7	70
B	4	15	1	15
C	1	8	0	0
C	3	12	5	60
D	2	2	0	0
D	3	6	4	24
D	4	7	(– 2)	– 14

Alternative solutions also exist Rs. 155.

46. A company has three plants in which it produces a standard product. It has four agencies in different parts of the country where this product is sold. The production cost varies from factory to factory and the selling price from market to market. The shipping cost per unit of the product from each plant to each of the agencies is known and is stable. The relevant data are given in the following table :

(a)

Plant	Weekly production capacity (Units)	Unit production cost (in Rs.)
1	400	18
2	300	24
3	800	20

(b) Shipping cost (in Rs.) per unit :

	To Agency 1	2	3	4
From Plant 1	2	5	7	3
2	8	4	6	2
3	3	4	4	5

(c)

Agency	Demand (units)	Selling price (Rs.)
1	300	32
2	400	35
3	300	31
4	500	36

Determine the optimum plan so as to maximize the profits. **(Delhi (MBA) Nov. 97)**

47. *ABC* Enterprises is having three plants manufacturing dry-cells, located at different locations. Production cost differs from plant to plant. There are five sales offices of the company located in different regions of the country. The sales prices can differ from region to region. The shipping cost from each plant to each sales office and other data are given by following table :

PRODUCTION DATA TABLE

Production cost per unit	Max. capacity in number of units	Plant number
20	150	1
22	200	2
18	125	3

Shipping costs :

	Sales Office 1	Sales Office 2	Sales Office 3	Sales Office 4	Sales Office 5
Plant 1	1	1	4	9	4
Plant 2	9	7	8	3	3
Plant 3	4	5	3	2	7

Demand & Sales prices

Demand	80	100	75	45	125
Sales Price	30	32	31	34	29

Find the production and distribution schedule most profitable to the company. **(C.A., Nov. 98)**

[**Hint :** Convert into loss matrix by subtracting each element from largest element 14. Add dummy column with demand 50.

Ans.

Plant	Sales Office	Units	Profit	Total Profit (Rs.)
P_1	S_1	50	9	450
P_1	S_2	100	11	1100
P_2	S_4	25	9	425
P_2	S_5	125	1	125
P_3	S_1	30	8	240
P_3	S_3	75	10	750
P_3	S_4	20	14	280
Total				3170

48. *XYZ* Co. has provided the following data seeking your advice on optimum investment strategy :

Investment made at the beginning of year	Net return data (in Paise) of selected investments *P*	*Q*	*R*	*S*	Amount available (lacs)
1	95	80	70	60	70
2	75	65	60	50	40
3	70	45	50	40	90
4	60	40	40	30	30
Maximum Investment (Lacs)	40	50	50	50	—

The following additional information are also provided :

— *P, Q, R* and *S* represent the selected investments.
— The company has decided to have four-year investment plan.
— The policy of the company is that amount invested in any year will remain so until the end of the fourth year.
— The values (paise) in the table represent net return on investment of one rupee till the end of the planning horizon. (For example,I a rupee invested in investment *P* at the beginning of year 1 will grow to Rs. 1.95 by the end of the fourth year, yielding) a return of 95 paise.

using the above, determine the optimum investment strategy. **(C.A., Nov. 96)**

[**Ans.** The optimum allocations given below.

Year	Investments (Rs.)	Net return (Rs. Lacks)
1	40 lacks to P, 30 lacs to Q	$0{\cdot}95 \times 40 = 38,\ 0{\cdot}80 \times 30 = 24$
2	20 lacks to Q, 20 lacks to R	$0{\cdot}65 \times 20 = 13,\ 0{\cdot}60 \times 20 = 12$
3	40 lacks to R, 50 lacks to S	$0{\cdot}50 \times 40 = 20,\ 0{\cdot}40 \times 50 = 20$
4	50 lacks to S	$0{\cdot}30 \times 10 = 3.$

49. A company has three factories, *A, B* and *C* located in different parts of the country. It has four supply points : *D, E, F* and *G*, also located in different parts of the country. Monthly factory production capacities are 2500, 3000 and 4000 units respectively with regular production. If overtime facilities are utilized, capacities of factories *A* and *B* can be increased by 20 per cent and 25 per cent additional cost of overtime production is 40 paise and 50 paise in factory *A* and *B* respectively. The current supply-point requirements are 2000, 2250, 2750 and 3000 units respectively. Unit production and transportation costs (in Rs.) from factories to supply points are as follows :

Factory	Supply points *D*	*E*	*F*	*G*
A	1·10	1·30	1·70	1·40
B	1·60	1·80	1·40	1·00
C	2·10	2·40	1·30	1·00

Determine the optimum distribution for this company to minimize total costs. **(Delhi (MBA) Dec. 96)**

50. *ABC* company wishes to develop a monthly production schedule for the next three months. Depending upon the sales commitments, the company can either keep the production constant, allowing fluctuations in inventory, or inventories can be maintained at a constant level, with fluctuating production. Fluctuating inventories result in inventory carrying cost of Rs. 2 per unit. If the company fails to fulfill its sales commitment, it occurs a shortage cost of Rs. 4 per unit per month. Assume that the shortages are back ordered. The production capacities for the next three months are shown below :

Month	Production Capacity		
	Regular	Overtime	Sales
1	50	30	60
2	50	0	120
3	60	50	40

Determine the optimum production schedule. **(Delhi (MBA) April 98)**

51. The following information is available concerning the operations of the *XYZ* manufacturing company :

	Month 1	Month 2
Units on order	800	1,400
Production capacity.		
Regular time	920	920
Overtime	250	250
Excess cost/unit-overtime	Rs. 1.25	Rs. 1.25
Storage cost/unit	Rs. 0.50	Rs. 0.50

Formulate and solve the above problem as transportation problem. **(Delhi (MBA) Nov. 95)**

52. Stronghold Construction Company is interested in taking loans from banks for some of its projects P,Q,R,S and T. The rates of interest and the landing capacity differ from bank to bank. All these projects are to be completed. The relevant details are provided in the following table :

Bank	Interest rate in percentage for project					Maximum Credit
	P	Q	R	S	T	(Rs.' 000)
Private Bank	20	18	18	17	17	Any amount
Nationalised Bank	16	16	16	15	16	400
Co-operative Bank	15	15	15	13	14	250
Amount required (Rs.'000)	200	150	200	125	75	

Assuming the role of consultant adviser of the company as to how it should take the loans so that the total interest payable will be the least. Are there alternate optimum solutions ? If so, indicate one such solution. **[Delhi (MCI) 2000]**

53. A baking firm can produce a speciality bread in either of its two plants as follows :

Plant	Production Capacity Loaves	Production cost Rs./Loaf
A	2,500	2·30
B	2,100	2·50

Four restaurant chains are willing to purchase this bread; their demand and the prices they are willing to pay are as follows :

Chain	Maximum Demand Loaves	Price Offered Rs./Loaf
1	1,800	3·90
2	2,300	3·70
3	550	4·00
4	1,750	3·60

The cost in paise of shipping a loaf from a plant to a restaurant chain is

Plant	Chain 1	Chain 2	Chain 3	Chain 4
A	6	8	11	9
B	12	6	8	5

Determining a delivery schedule for the baking firm that will maximize its profit from this bread. Also, write the dual of this transportation problem and use it for checking the optimal solution to the given problem. **[Delhi (MBA) 2000]**

54. A manufacturer has distribution centres at Delhi Kolkata and Chennai. These centres have available 30, 50 and 70 units of product. His four retail outlets require the following number of units.

A, 30; B, 20; C, 60; D, 40;

The transportation cost per unit in rupees between each centre and outlet is given in the following table :

Distribution Centres	Retail Outlets A	B	C	D
Delhi	10	7	3	6
Kolkata	1	6	7	3
Chennai	7	4	5	3

Determine the minimum transportation cost. **[IAS (Main) 2001]**

55. A company has factories at four different places which supply to warehouses *A, B, C, D* and *E*. Monthly factory capacities are 200, 175, 150 and 325 respectively. Unit shipping costs in rupees are given below :

		Warehouses A	B	C	D	E
Factories	1	13	—	31	8	20
	2	14	9	17	6	10
	3	25	11	12	17	15
	4	10	21	13	—	17

Shipping from 1 to B and from 4 to D is not possible. Determine the optimum distribution to minimize the shipping are cost. **[AIMS (BE Ind.) Bangl. 2002]**

56. The following information is available concerning the operations of the xyz manufacturing company :

Type	Month 1	Month 2
Units on order	800	1,400
Production capacity regular time	920	920
Production capacity over time	250	250
Excess cost/unit (over time)	Rs. 1.25	Rs. 1.25
Storage cost/unit	Rs. 0.50	Rs. 0.50

Formulate and solve the above problem as transportation problem. **[Delhi (MBA) 2002]**

57. What is the total number of constraint equations in a general transportation model with *m* sources and *n* destinations ? How many of these are independent ? Justify your answer. Name the methods for obtaining an initial basic solution. Which is the best one, and why ? Also describe it. **[Meerut (OR) 2003]**

58. Solve the following transportation problem by finding the initial solution by using VAM :

Source \ Destination	D_1	D_2	D_3	Supply
S_1	8	7	4	60
S_2	3	8	9	70
S_3	11	3	5	80
Demand	50	80	80	210

[JNTU (MCA III) 2004]

59. Consider four basis of operation B_i and three targets T_j. The tons of bombs per aircraft from any base that can be delivered to any target are given in the side table :

The daily sortie capacity of each of the four basis is 150 sorties per day. The daily requirement in sorties over each individual target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all three targets. **[JNTU (Mech. & Prod.) May 2004]**

	T_1	T_2	T_3
B_1	8	8	5
B_2	6	6	6
B_3	10	8	4
B_4	8	6	10

OBJECTIVE QUESTIONS

1. The initial solution of a transportation problem can be obtained by applying any known method. However, the only condition is that
(a) the solution be optimal. (b) the rim conditions are satisfied.
(c) the solution not be degenerate. (d) all of the above.

2. The dummy source or destination in a transportation problem is added to
(a) satisfy rim conditions. (b) prevent solution from becoming degenerate.
(c) ensure that total cost does not exceed a limit. (d) none of the above.

3. The occurrence of degeneracy while solving a transportation problem means that
(a) total supply equals total demand. (b) the solution so obtained is not feasible.
(c) the few allocations become negative. (d) none of the above.

4. An alternative optinal solution to a minimization transportation problem exists whenever opportunity cost corresponding to unused route of transportation is:
(a) positive and greater than zero. (b) positive with at least one is equal to zero.
(c) negative with at least one is equal to zero. (d) none of the above.

5. One disadvantage of using North-West Corner Rule to find initial solution to the transportation problem is that
(a) it is complicated to use. (b) it does not take into account the cost of transportation.
(c) it leads to a degenerate initial solution. (d) all of the above.

6. The solution to a transportation problem with m - rows (supplies) and n- columns (destinations) is feasible if number of positive allocations are
(a) $m + n$. (b) $m \times n$. (c) $m + n - 1$. (d) $m + n + 1$.

7. The number of non-negative variables in a basic feasible solution to a $m \times n$ transportation problem is :
(a) mn. (b) $m + n$. (c) $m + n + 1$. (d) $m + n - 1$.'

8. The calculation of opportunity cost in the MODI method is analogous to a
(a) $z_j - c_j$ value for non-basic variable columns in the simplex method.
(b) value of a variable in x_B-column of the simplex method.
(c) variable in the B-column in the simplex method. (d) none of the above.

9. An unoccupied cell in the transportation method is analogous to a
(a) $z_j - c_j$ value in the simplex method. (b) variable in the **B**-column in the simplex method.
(c) variable not in the **B**-column in the simplex method. (d) value in the X_B-column in the simplex method.

10. If we were to use opportunity cost value for an unused cell to test optimality, it should be
(a) equal to zero. (b) most negative number. (c) most positive number. (d) any value.

11. During an iteration while moving from one solution to the next, degeneracy may occur when
(a) the closed path indicates a diagonal move.
(b) two or more occupied cells are on the closed path but neither of them represents a corner of the path.
(c) two or more occupied cells on the closed path with minus sign are tied for lowest circled value.
(d) either of the above.

Answers

1. (b) **2.** (a) **3.** (b) **4.** (b) **5.** (b) **6.** (c) **7.** (d) **8.** (a) **9.** (c) **10.** (b) **11.** (c).

❋❋❋

12

ASSIGNMENT PROBLEMS

12.1. INTRODUCTION

As already discussed earlier, linear programming relates to the problems concerning distributions of various resourses (such as *money, machines, time* etc.), satisfying some constraints which can be algebraically represented as linear *equations/inequalities* so as to *maximize profit* or *minimize cost.* This chapter deals with a very interesting method called the *'Assignment Technique'* which is applicable to a class of very practical problems generally called *'Assignment problems'*.

The name *'Assignment Problem'* originates from the classical problems where the objective is to assign a number of origins (jobs) to the equal number of destinations (persons) at a minimum cost (or maximum profit). To examine the nature of assignment problem, ***suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degree of efficiency. Let c_{ij} be the cost (payment) if the ith person is assigned the jth job, the problem is to find an assignment (which job should be assigned to which person) so that the total cost for performing all jobs is minimum.*** Problems of this kind are known as *assignment problems.*

Table 12·1

Jobs

	1	2	...	j	...	n
1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}
2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}
Persons :	:	:		:		:
i	c_{i1}	c_{i2}	...	c_{ij}	...	c_{in}
:	:	:		:		:
n	c_{n1}	c_{n2}	...	c_{nj}	...	c_{nn}

Further, such types of problems may consist of assigning men to offices, classes to rooms, drivers to trucks, trucks to delivery routes, or problems to research teams, etc. The assignment problem can be stated in the form of $n \times n$ cost-matrix $[c_{ij}]$ of real number as given in Table 12·1 .

Q. 1. Define Assignment Problem.

2. What is an assignment problem ? **[IGNOU 2001, 99, 97, 96]**

12.2. MATHEMATICAL FORMULATION OF ASSIGNMENT PROBLEM

Mathematically, the assignment problem can be stated as :

$$\textit{Minimize the total cost} : z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}, \; i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, n \qquad \ldots(12\cdot1)$$

subject to restrictions of the form :

$$x_{ij} = \begin{cases} 1 & \text{if } ith \text{ person is assigned } jth \text{ job} \\ 0 & \text{if } not \end{cases} \qquad \ldots(12\cdot2)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \textit{(one job is done by the ith person, } i = 1, 2, \ldots, n) \qquad \ldots(12\cdot3)$$

and $$\sum_{i=1}^{n} x_{ij} = 1 \qquad \textit{(only one person should be assigned the jth job, } j = 1, 2, \ldots, n) \qquad \ldots(12\cdot4)$$

where x_{ij} denotes that jth job is to be assigned to the ith person.

This special structure of assignment problem allows a more convenient method of solution in comparison to *simplex method.*

12·2-1. Assignment Problem as Special Case of Transportation Problem

The assignment problem (as defined in previous chapter) is seen to be the special case of transportation problem when each origin is associated with one and only one destination. In such a case, $m = n$ and the numerical evaluations of such association are called 'effectiveness' instead of 'transportation costs'. Mathematically, all a_i and b_j are unity, and each x_{ij} is limited to one of the two values 0 and 1. In such circumstances, exactly n of the x_{ij} can be non-zero (*i.e.* unity), one for each origin and one for each destination.

Q. 1. Give the mathematical formulation of an assignment problem.
[JNTU (B. Tech.) 2000, 03; Meerut (Stat.) 98; Rewa (M.P.) 93 ; Meerut (IPM) 90]

2. Explain the conceptual justification that an assignment problem can be viewd as a linear programming problem. **[VTU 2002]**
3. What is a Transportation Problem ? **[Meerut 2002; IGNOU 99, 96]**
4. Give a mathematical formulation of the transportation and the simplex methods. What are the differences in the nature of problems that can be solved by these methods.
5. Give the mathematical formulation of transportation problem. How does it differ from an assignment problem ?
[VTU (BE Common) Feb. 2002; Meerut 2002; Madurai B.Sc. (Compu. Sc.) 92; Bharathidasan B.Sc. (Math) 90]
6. What is meant by a classical transportation problem ? What is its mathematical formulation ? Explain briefly your symbols.

12.3. FUNDAMENTAL THEOREMS OF ASSIGNMENT PROBLEM

The solution to an assignment problem is fundamentally based on the following two theorems.

Theorem 12.1. ***Reduction Theorem*** **:** *In an assignment problem, if we add (or subtract) a constant to every element of a row (or column) of the cost matrix $[c_{ij}]$, then an assignment plan that minimizes the total cost for the new cost matrix also minimizes the total cost for the original cost matrix.*

[Meerut (Maths) 2005, 96, 94, 93 P; Rewa (MP) 93]

Proof. Let $x_{ij} = X_{ij}$ minimizes the toal cost,

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad \ldots(12\cdot5)$$

over all x_{ij} such that $\quad x_{ij} \ge 0$ and $\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1$. $\qquad \ldots(12\cdot6)$

It is to be shown that the assignment $x_{ij} = X_{ij}$ also minimizes the new total cost

$$z' = \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - u_i - v_j)\, x_{ij}$$

for all $i, j = 1, 2, \ldots, n$, where u_i and v_j are constants subtracted from ith row and jth column of the cost matrix $[c_{ij}]$. To prove this, it may be written as

$$z' = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} v_j \sum_{i=1}^{n} x_{ij}.$$

Using equations (12·5) and (12·6), we get

$$z' = z - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j .$$

Since terms that are subtracted from z to give z' are independent of x_{ij}'s, it follows that z' is minimized whenever z is minimized, and conversely.

This completes the proof of the theorem.

Alternative statement of reduction theorem : *If $x_{ij} = X_{ij}$ minimizes $z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}$ over all $x_{ij} = 0$ or 1 such that*

$$\sum_{i=1}^{n} x_{ij} = 1 , \sum_{j=1}^{n} x_{ij} = 1 \text{ , then } x_{ij} = X_{ij}$$

also minimizes $$z' = \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij} x_{ij} \text{ , where } c'_{ij} = c_{ij} - u_i - v_j \text{ for } i, j = 1, 2, \ldots, n .$$

and where u_i and v_j are some real numbers.

Corollary. *If (x_{ij}), $i = 1, 2....,n$; $j = 1, 2, ..., n$ is an optimal solution for an assignment problem with cost (c_{ij}), then it is also optimal for the problem with cost (c_{ij}') when*

$c'_{ij} = c_{ij}$ for $i, j = 1, 2, \ldots , n ; j \neq k$

$c'_{ik} = c_{ik} - A$, *where A is a constant.*

Proof. We have

$$z' = \sum_i \sum_j c'_{ij} x_{ij} = \sum_i \left(\sum_{j \neq k} c'_{ij} + c'_{ik} \right) x_{ij} = \sum_i \left(\sum_{j \neq k} c_{ij} + c_{ik} - A \right) x_{ij} = \sum \sum c_{ij} x_{ij} - A \sum_i x_{ij}$$

$$= z - A \qquad \left(\text{since } \sum_i x_{ij} = 1 \right)$$

Thus if (x_{ij}) minimizes z so will it z' .

Theorem 12.2. *In an assignment problem with cost (c_{ij}) , if all $c_{ij} \geq 0$ then a feasible solution (x_{ij}) which satisfies $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} = 0$, is optimal for the problem.*

Proof. Since all $c_{ij} \geq 0$ and all $x_{ij} \geq 0$, the objective function $z = \Sigma \Sigma c_{ij} x_{ij}$ cannot be negative. The minimum possible value that z can attain is 0. Thus, any feasible solution (x_{ij}) that satisfies $\Sigma \Sigma c_{ij} x_{ij} = 0$ will be optimal.

Theorem 12.3. ***(König Theorem).*** *Let P be the set of 0 elements of a matrix C. Then the maximum number of 0's that can be selected in P such that no row or column of C contains more than one such 0 is equal to the minimum number of lines covering all the elements of P.*

Proof is beyond the scope of the book.

Corollary. *The maximal subset of P provides an optimal assignment when the minimum number of lines to cover all the elements of P is equal to the order of C.*

Proof. Left as an exercise for the reader.

Q. 1. Explain how an assignment problem can be treated as a linear programming problem. Show that the optimal solution to the assignment problem remains the same if a constant is added to or subtracted from any row or column of the cost matrix.

2. If $b_{ij} = c_{ij} - u_i - v_j$ $(i, j = 1, 1, 2, \ldots , n)$ where u_i and v_j are constants, then show that an optimal solution of the assignment problem with cost matrix $B = (b_{ij})$ is also an optimal solution of the assignment problem with cost matrix $C = (c_{ij})$.

[Delhi B.Sc. (Math.) 90]

12.4. HUNGARIAN METHOD FOR ASSIGNMENT PROBLEM

The solution technique of assignment problems can be easily explained by the following practical examples.

Example 1. *A department head has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours ?* **[JNTU 2002, 2000; Tamil. (ERODE) 97; IAS (Main) 93; Kerala B.Sc. (Math.) 91; Meerut (Stat.) 90; Kalicut B. Tech 90]**

Table 12·2

Subordinates

		I	II	III	IV
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution. To understand the problem initially, step by step solution procedure is necessary.

Step 1. Subtracting the smallest element in each row from every element of that row, we get the reduced matrix [*Table* 12·3]

Step 2. Next subtract the smallest element in each column from every element of that column to get the second reduced matrix [*Table* 12·4]

Table 12·3

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Table 12·4

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Step 3. Now, test whether it is possible to make an assignment using only *zeros.* If it is possible, the assignment must be optimal by *Theorem 12.2* of Section 12·3 . Zero assignment is possible in *Table* 12·4 as follows :

(a) Starting with *row* 1 of the matrix (*Table* 12·4), examine the rows one by one until a row containing exactly *single zero* element is found. Then an experimental assignment (indicated by □) is marked to that cell. Now cross all other zeros in the *column* in which the assignment has been made. This eliminates the possibility of marking further assignments in that column. The illustration of this procedure is shown in *Table* 12·5a.

Table 12·5a

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	0	3	0
D	9	12	14	$\boxed{0}$

Table 12·5b

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	0
D	9	12	14	$\boxed{0}$

(b) When the set of rows has been completely examined, an identical procedure is applied sucessively to columns. Starting with *column* 1, examine all columns until a *column* containing exactly one zero is found. Then make an experimental assignment in that position and cross other zeros in the *row* in which the assignment has been made.

Continue these successive operations on rows and columns until all zeros have been either assigned or crossed-out. At this stage, re-examine rows. It is found that no additional assignments are possible. Thus, the complete '*zero assignment*' is given by A → I , B → III, C → II , D → IV as mentioned in *Table 12·5b.* According to *Theorem* 12.1, this assignment is also optimal for the original matrix (*Table 12·2*). Now compute the minimum total man-hours as follows :

Optimal assignment	:	A—I	B—III	C—II	D—IV	
Man-hour	:	8	4	19	10	(Total 41 hours.)

Now the question arises : *what would be further steps if the complete optimal assignment after applying Step 3 is not obtained ?* Such difficulty will arise whenever all zeros of any row or column are crossed-out. Following example will make the procedure clear.

Example 2. *A car hire company has one car at each of five depots a ,b , c, d and e. A customer requires a car in each town, namely A, B, C, D, and E. Distance (in kms) between depots (origins) and towns (destinations) are given in the following distance matrix :*

Table 12·6

	a	*b*	*c*	*d*	*e*
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How should cars be assigned to customers so as to minimize the distance travelled ? **[Delhi (MBA) 2002, 99]**

Solution. Applying *Step 1* and *Step 2* as explained in *Example 1* we get the *Table* 12·7 .

Table 12·7

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Step 3. Row 1 has a single zero in *column* 2. Make an assignment by putting a square '□' around it, and delete other zero (if any) in column 2 by marking '×'.

Table 12·8 a

30	$\boxed{0}$	35	30	15
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$
30	$\cancel{0}$	35	30	20
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5
20	$\cancel{0}$	25	15	15

Now, *column* 1 has a single zero in *row* 4. Make an assignment by putting '□' and cross the other zeros which is not yet crossed. Column 3 has a single zero in *row* 2, make an assignment and delete the other zeros which are uncrossed.

It is observed that there are no remaining zeros ; and *row 3, row 5, column 4,* and *column 5* each has no assignment. Therefore, desired solution cannot be obtained at this stage. we now, proceed to following important steps.

Step 4. Draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. It should, however, be observed that (in all $n \times n$ matrices) less than n lines will cover zeros only when there is no solution among them. Conversely, if minimum number of lines is n , there is a solution. Following systematic procedure may help us to draw the minimum set of lines :

1. For simplicity, first make the *Table* 12·8*a* again and name it at *Table* 12·8 b.

Table 12·8 b

		L_1				
	30	$\boxed{0}$	35	30	15	✓,④
L_2	15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
	30	$\cancel{0}$	35	30	20	✓ ①
L_3	$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
	20	$\cancel{0}$	25	15	15	✓ ②
		③✓				

2. Mark (√) *row 3* and *row 5* as they are having no assignments and *column 2* as having zeros in the marked rows 3 and 5.

3. Mark ($\sqrt{}$) *row 1* because this row contains assignment in the marked *column 2.* No further rows or columns will be required to mark during this procedure.
4. Now start drawing required lines as follows ;
   First draw line (L_1) through *marked* column 2. Then draw lines(L_2 and L_3) through *unmarked* rows (2 and 4) having largest number (2) of uncovered zeros (since no zero is left uncovered, the required lines will be (L_1 , L_2 and L_3) .

Step 5. In this step,

(i) first select the smallest element, say x, among all *uncovered* elements of the *Table* 12·8b [as a result of *step* 4] and

(ii) then subtract this value x from all values in the matrix not covered by lines and add x to all those values that lie at the intersection of any two of the lines L_1 , L_2 and L_3 . (Justification of this rule is given on the next page).

After applying these two rules, we find $x = 15$, and a new matrix is obtained as given in *Table 12·9*.

Table 12·9

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

Step 6. Now re-apply the test of *Step 3* to obtain the desired solution. Therefore, proceeding exactly in the same manner as in *step 3*, obtain the final *Table* 12·10 .

Table 12·10

15	$\cancel{0}$	20	15	$\boxed{0}$
15	15	$\boxed{0}$	10	$\cancel{0}$
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	20	$\cancel{0}$	5
5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

It is observed that there are no remaining zeros, and every row (column) has an assignment. Since no two assignments are in the same column (they cannot be, if the procedure has been correctly followed), the 'zero assignment' is the required solution.

From original matrix (*Table* 12·6), the minimum distance assignment is given by

Route	A–e	B–c	C–b	D–a	E–d	Total Distance Travelled
Distance (*Kms.*)	200	130	110	50	80	570 Kms.

Note. Table 12·10 may be obtained very quickly if we first apply ***Step 2*** and then ***Step 1*** in the original Table 12·6 ;

Justification of rules used above in step 5 :

Justification of rules we have used in *Step* 5 is based on the following two facts :

(i) The relative cost of assigning ith facility to jth job is not changed by the subtraction of a constant either from a column or from a row of the original effectiveness matrix.

(ii) An optimal assignment exists if the total reduced cost of the assignment is zero. This is the case when the minimum number of lines necessary to cover all zeros is equal to the order of the matrix. If however, it is less than n further reduction of the effectiveness matrix has to be undertaken.

The underlying logic can be explained with the help of *Table 12·8(b)* in which only 3 ($= n - 2$) lines can be drawn. Here an optimal assignment is not possible. So further reduction is necessary.

Further reduction is made by subtracting the smallest non-zero element 15 from all elements of the matrix *Table 12.8(b).* This gives the following matrix :

This matrix contains negative values. Since the objective is to obtain an assignment with reduced cost of zero, the negative numbers must be eliminated. This can be done by adding 15 to only those rows and columns which are covered by three *lines* (L_1, L_2, L_3) as shown above. In doing so the following change is noted.

		L_1			
	15	− 15	20	15	0
L_2	0	−15	15	− 5	− 15
	15	− 15	20	15	0
L_3	15	−15	5	− 15	− 10
	5	− 15	10	0	0

This table is exactly the same as Table 12·9 . In fact, all this is the result of adding the least non-zero element at the intersections; and subtracting from all uncovered elements, and leaving the other elements unchanged.

		L_1			
	15	(−15+15)	20	15	0
L_2	(0 + 15)	[(−15+15)+15]	(− 15 + 15)	(− 15 + 15)	(− 15 + 15)
	15	(−15+15)	20	15	5
L_3	(− 15 + 15)	[(−15+15)+15]	(5 + 15)	(− 5 + 15)	(− 10 + 15)
	5	(−15+15)	10	0	0

Q. 1. Show that the procedure of subtracting the minimum elements not covered by any line, from all the uncovered elements and adding the same element to all the elements lying at the intersection of two lines results in a matrix with the same optimal assignments as the original matrix. **[Meerut M.Sc. (Math.) 92, 90; Jodhpur M.Sc. (Math) 92]**

2. State the assignment problem. Describe a method of drawing minimum number of lines in the context of assignment problem. Name the method.

3. Describe any method for solving an assignment problem. **[Delhi B.Sc. (Math.) 93]**

4. Show that the lines drawn in the assignment algorithm pass through all the zeros and have the property that every line passes through one and only one assignment.

5. Show that in an assignment, if we multiply each element of the effectiveness (cost) matrix by some fixed number, then the optimal solution remains unchanged.

11·4-1. Assignment Algorithm (Hungarian Assignment Method)

Various steps of the computational procedure for obtaining an optimal assignment may be summarized as follows : **[JNTU (MCA III) 2004]**

Step 1. Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows.

Step 2. Further, modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus obtain the *first modified matrix.*

Step 3. Then, draw the minimum number of horizontal and vertical lines to cover all the zeros in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities :

(i) If $N = n$, the number of rows (columns) of given matrix, then an optimal assignment can be made. So make the zero assignment to get the required solution.

(ii) If $N < n$, then proceed to *step 4.*

Step 4. Determine the smallest element in the matrix, not covered by the N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the *second modified matrix* is obtained.

Step 5. Again repeat ***Steps 3 and 4*** until minimum number of lines become equal to the number of rows (columns) of the given matrix *i.e.,* $N = n$.

Step 6. ***(To make zero-assignment).*** Examine the rows successively until a row-wise exactly single zero is found, mark this zero by '□' to make the assignment. Then, mark a cross (×) over all zeros if lying in the column of the marked '□' zero, showing that they cannot be considered for future assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for columns also.

Step 7. Repeat the ***Step 6*** successively until one of the following situations arise :
(i) if no unmarked zero is left, then the process ends ; or
(ii) if there lie more than one of the unmarked zeros in any column or row, then mark '□' one of the unmarked zeros arbitrarily and mark a cross in the cells of remaining zeros in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Step 8. Thus exactly one marked '□' zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked '□' *zeros* will give the optimal assignment.

12·4-2 A Rule to Draw Minimum Number of Lines

A very convenient rule of drawing minimum number of lines to cover all the 0's of the reduced matrix is given in the follwing steps :

***Step* 1.** Tick (√) rows that do not have any marked (□) zero.
***Step* 2.** Tick (√) columns having marked (□) zeros or otherwise in ticked rows.
***Step* 3.** Tick (√) rows having marked 0's in ticked columns.
***Step* 4.** Repeat *steps 2 and 3* until the chain of ticking is complete.
***Step* 5.** Draw lines through all unticked rows and ticked columns.
This will give us the minimal system of lines.

Q. 1. Give an algorithm to solve an 'Assignment Problem'. **[IGNOU 2001, 99, 97, 96; IAS (Maths) 88]**
2. Write a short note on 'Assignment Problem'.
3. Explain the Humgarian method to solve an assignment problem. **[Meerut (OR) 2003, 02; VTU 2002]**

12.5. MORE ILLUSTRATIVE EXAMPLES

Example 3. *Solve the assignment problem represented by the following matrix.* (*Table* 12·11).
[IAS (Maths.) 96; Meerut (B.Sc.) 90]

Table 12·11

	a	*b*	*c*	*d*	*e*	*f*
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	32
E	36	11	57	22	25	18
F	3	56	53	31	17	28

Solution. Step 1. Subtract the smallest element in each row from every element in that row to get the reduced matrix. (*Table 12·12*).

Table 12·12

0	13	49	2	10	18
0	35	29	7	20	5
13	0	63	9	17	5
47	15	0	22	12	5
25	0	46	11	14	7
0	53	50	28	14	25

Step 2. Subtract the smallest element in each column from every element in that column to get the second reduced matrix. (*Table* 12·13).

Step 3. Make the 'zero-assignments' in usual manner. Since *row 2* and *column 5 of Table 12.13* have no assignments, go to next step.

Table 12·13

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13
$\cancel{0}$	35	29	5	10	$\cancel{0}$
13	$\cancel{0}$	63	7	7	$\boxed{0}$
47	15	$\boxed{0}$	20	2	$\cancel{0}$
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

Step 4. Draw minimum number of lines to cover all zeros at least once. To do so, first mark (√) *row 2* as having no assignment and *columns* (*1* and *6*) as having zeros in row 2. Next, mark (√) the rows (3 and 6) as these two rows contain assignment in the marked columns (*1* and *6*).

Now draw lines L_1 and L_2 through each marked column (1 and 6), respectively. Then draw line L_3 through unmarked row 1 and line L_4 through unmarked column 2 (both having two uncovered zeros). Draw one more line L_5 either through unmarked row 4 or unmarked column 3. This way the minimum set of five lines (which is less than six) is obtained.

Table 12·14

	L_1	L_4				L_2	
L_3	0	13	49	$\boxed{0}$	$\cancel{0}$	13	
	0	35	29	5	10	0	√ 1
	13	0	63	7	7	$\boxed{0}$	√ 5
L_5	47	15	$\boxed{0}$	20	2	0	
	25	$\boxed{0}$	46	9	4	2	√ 7
	$\boxed{0}$	53	50	26	4	20	√ 4
	√ 2	√ 6				√ 3	

Step 5. The smallest element among all *uncovered* elements of *Table* 12·14 is 4. *Subtract* this value 4 from all values in *Table* 12·14 *not covered by lines*, and *add* 4 to all those values that lie at the intersection of the lines L_1, L_2, L_3, L_4 and L_5. Thus a new matrix (*Table 12·15a*) is obtained.

Step 6. Repeating the *Step 3*, make the 'zero assignments' as shown in *Table 12·15a*.

Table 12·15 a

	a	b	c	d	e	f
A	4	17	49	$\boxed{0}$	$\cancel{0}$	17
B	$\boxed{0}$	35	25	1	6	$\cancel{0}$
C	13	$\cancel{0}$	59	3	3	$\boxed{0}$
D	51	19	$\boxed{0}$	20	2	4
E	25	$\boxed{0}$	42	5	$\cancel{0}$	2
F	$\cancel{0}$	53	46	22	$\boxed{0}$	20

Here, it is also important that an assignment problem may have more than one solution. The other solution is shown in *Table 12·15b*.

Table 12·15 b

	a	b	c	d	e	f
A	4	17	49	$\boxed{0}$	$\cancel{0}$	17
B	$\cancel{0}$	35	25	1	6	$\boxed{0}$
C	13	$\boxed{0}$	59	3	3	$\cancel{0}$
D	51	19	$\boxed{0}$	20	2	4
E	25	$\cancel{0}$	42	5	$\boxed{0}$	2
F	$\boxed{0}$	53	46	22	$\cancel{0}$	20

These two solutions are :

(i) A → d, B → a, C → f, D → c, E → b and F → e ; (ii) A → d, B → f, C → b, D → c, E → e and F → a. with minimum cost z = **Rs. 142**.

Example 4. ***(Alternative Solutions).*** *Solve the minimal assignment problem whose effectiveness matrix is given by Table* 12·16. **[Meerut 2002]**

Table 12·16

	1	2	3	4
I	2	3	4	5
II	4	5	6	7
III	7	8	9	8
IV	3	5	8	4

Solution. Step 1. For each row in the matrix, subtract the smallest element in the row from each element in that row to get reduced matrix (*Table 12·17*)

Step 2. For each column in the reduced matrix, subtract the smallest element in the column from each element of that column to get the second reduced matrix *(Table 12·18)*

Table 12·17

0	1	2	3
0	1	2	3
0	1	2	1
0	2	5	1

Table 12·18

0	0	0	2
0	0	0	2
0	0	0	0
0	1	3	0

Step 3. Since single zeros neither exist in columns nor in rows, it is usually easy to make zero assignments. While examining rows successively, it is observed that *row 4* has two zeros in both the cells (4, 1) and (4, 4). Now, arbitrarily make an experimental assignment (indicated by □) to one of these two cells, say (4,1) and cross other zeros in *row 4* and *column 1*. *Tables 12·19*a, *12·19*b and *12·19*c show the necessary steps for reaching the optimal assignment : I → 2 , II → 3 , III → 4 , IV → 1 .

Table 12·19 a

	1	2	3	4
I	$\cancel{0}$	0	0	2
II	$\cancel{0}$	0	0	2
III	$\cancel{0}$	0	0	0
IV	$\boxed{0}$	1	3	$\cancel{0}$

Table 12·19 b

	1	2	3	4
I	$\cancel{0}$	0	0	2
II	$\cancel{0}$	0	0	2
III	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
IV	$\boxed{0}$	1	3	$\cancel{0}$

Table 12·19 c

	1	2	3	4
I	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
II	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
III	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
IV	$\boxed{0}$	1	3	$\cancel{0}$

Following other optimal assignments are also possible in this example.

$$\left\{\begin{array}{l} \text{I} \to 1\,,\ \text{II} \to 2\,,\ \text{III} \to 3\,,\ \text{IV} \to 4,\ \text{I} \to 3\,,\ \text{II} \to 2\,,\ \text{III} \to 1\,,\ \text{IV} \to 4 \\ \text{I} \to 3\,,\ \text{II} \to 2\,,\ \text{III} \to 4\,,\ \text{IV} \to 1,\ \ \text{I} \to 2\,,\ \text{II} \to 3\,,\ \text{III} \to 1\,,\ \text{IV} \to 4 \end{array}\right\} \text{(each having the cost 20)}$$

Example 5. (***A Typical Problem***) *An air-line operating seven days a week has time-table shown below. Crews must have a minimum layover (rest) time of 5 hrs between flights. Obtain the pair of flights that minimizes layover time away from home. For any given pair, the crew will be based at the city that results in the smaller layover. For each pair, mention the town where the crews should be based.*

Delhi-Jaipur			Jaipur-Delhi		
Flight No.	Depart	Arrive	Flight No.	Depart	Arrive
1	7.00 A.M	8.00 *A.M.*	101	8.00 A.M.	9.15 A.M.
2	8.00 A.M.	9.00 *A.M.*	102	8.30 A.M.	9.45 A.M.
3	1.30 P.M.	2.30 *P.M.*	103	12.00 Noon	1.15 P.M.
4	6.30 P.M.	7.30 *P.M.*	104	5.30 P.M.	6.45 P.M.

[Meerut 2002, 99, 98, 96BP]

Solution. Step 1. Construct the table for layover times between flights when crew is based *at Delhi.* For simplicity, consider 15 minutes = 1 unit.

Table 12·20 . Layover times when crew based at Delhi

Flights	101	102	103	104
1	96	98	112	38
2	92	94	108	34
3	70	72	86	108
4	50	52	66	88

Since the crew have a minimum layover of 5 hrs between flights, the layover time between flights 1 and 101 will be 24 hrs (96 units) from 8.00 *A.M.* to 8.00 *A.M.* next day.

Likewise, calculate as follows :

Flight No.	Layover Times	No. of Units (1 hr. = 4 units)
$1 \rightarrow 102$	8 *A.M.* – 8.30 *A.M.* = 24 hrs 30 min	98
$1 \rightarrow 103$	8 *A.M* – 12 *Noon* = 28 *hrs*	112
$1 \rightarrow 104$	8 *A.M.* – 5.30 *P.M.* = 9 hrs 30 min	38
$2 \rightarrow 101$	9.00 *A.M.* – 8 *A.M.* = 23 hrs	92

and so on.

Similarly, layover times for other pair of flights can also be calculated as shown in *Table 12·20*.

Step 2. Similarly, construct the table for layover times between flights when crew is based at *Jaipur*.

Table 12·21. Layover times when crew based at Jaipur

Flights $\rightarrow$ / $\downarrow$	101	102	103	104
1	87	85	71	49
2	91	89	75	53
3	113	111	97	75
4	37	35	21	95

Since the plane arrives *Delhi* at 9·15 *A.M.* by flight number 101 and again depart to *Jaipur* at 7·00 *A.M.* by flight number 1, the layover time is obviously 21 hrs 45 min (*i.e.* 87 units). Similarly, layover times between other pairs of flight can also be computed as shown in *Table 12·21*.

Step 3. Construct the table for smaller layover times between flights with the help of *Tables 12·20* and *12·21*. Layover times marked '*' denote that the crew is based at *Jaipur*. Thus *Table 12·22* is obtained.

Table 12·22. Smaller layover times

	101	102	103	104
1	87*	85*	71*	38
2	91*	89*	75*	34
3	70	72	86	75*
4	37*	35*	21*	88

Step 4. Finally applying the assignment technique in the usual manner, we get the *Table 12·23*.

Table. 12.23

	101	102	103	104
1	4*	~~0~~	$\boxed{0^*}$	~~0~~
2	12*	8*	8*	$\boxed{0}$
3	$\boxed{0}$	~~0~~	28	50*
4	4*	$\boxed{0^*}$	~~0~~*	100

From *Table* 12·23 the optimal assignments are : (3 – 101) , (4 – 102)* , (1 – 103)* , (2 – 104) which gives the minimum layover time of 52 hr. 30 min.

Example 6. *A certain equipment needs five repair jobs which have to be assigned to five machines. The estimated time (in hours) that each mechanic requires to complete the repair job is given in the following table :*

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	7	5	9	8	11
M_2	9	12	7	11	10
M_3	8	5	4	6	9
M_4	7	3	6	9	5
M_5	4	6	7	5	11

Assuming that each mechanic can be assigned to only one job, determine the minimum time assignment.

(Rajasthan (M. Com.) 97)

Solution. Step 1. Subtracting the smallest element of each row from all the elements of that row and then subtracting the smallest element of each column from all the elements of that column, we get the reduced matrix as shown in Table 12.24 and Table 12.25 respectively.

Table 12.24

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	0	4	3	6
M_2	2	5	0	4	3
M_3	4	1	0	2	5
M_4	4	0	3	6	2
M_5	0	2	3	1	7

Table 12.25

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	0	4	2	4
M_2	2	5	0	3	1
M_3	4	1	0	1	3
M_4	4	0	3	5	0
M_5	0	2	3	0	5

Step 2. Now we attempt to make a complete set of assignments using only a single zero element in each row or column. Since row M_1 contains only single zero, the assignment is made in the cell (M_1, J_2) and the zero appearing in the corresponding column J_2 is crossed out. Similarly, the assignment is made in the cell (M_2, J_3) and the zero appearing in the corresponding column J_4 is crossed out. Now row M_4 has only single zero, therefore the assignment is made in cell (M_4 , J_5). Since there are two zeros in row M_5 , we cannot make assignment in this row M_5 . Looking columnwise, we find that column J_1 has only single zero, therefore we make an assignment in cell (M_5 , J_1) and cross out the zero appearing in the corresponding row M_5 . The assignments so made are shown in table 12.26.

Table 12.26

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	$\boxed{0}$	4	2	4
M_2	2	5	$\boxed{0}$	3	1
M_3	4	1	$\cancel{0}$	1	3
M_4	4	$\cancel{0}$	3	5	$\boxed{0}$
M_5	$\boxed{0}$	2	3	$\cancel{0}$	5

Thus, it is possible to make only four of the five necessary assignments using the zero element position. We, therefore, create one more zero element by drawing the minimum number of horizontal and vertical lines. Usually the minimum number of lines to cover all the zeros can be obtained by inspection. However, we shall use the method given earlier in explaining the various steps. The various steps for drawing the minimum number of lines are :

(a) Mark the row M_3 which has no assignment.

(b) Mark column J_3 which has zero in the marked row M_3.

(c) Mark row M_2 which has assignment in marked column J_3 ,

(d) Repeat steps (a) and (b) until no more rows or columns can be marked.

(e) Draw the lines through unmarked rows and marked columns.

The minimum number of lines drawn are shown in table 12.27. It must be checked that the number of lines drawn are equal to the number of assignments made. But we require five assignments. To create one more zero, we examine the elements not covered by these lines and select the smallest element, *viz.,* 1 from among these uncovered lines.

Table 12.27

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	$\boxed{0}$	4	2	4
M_2	2	5	$\boxed{0}$	3	1
M_3	4	1	$\cancel{0}$	1	3
M_4	4	$\cancel{0}$	3	5	$\boxed{0}$
M_5	$\boxed{0}$	2	3	$\cancel{0}$	5

Subtract this smallest element 1 from all the uncovered elements and add it to the element where the two lines intersect. The reduced matrix so obtained is shown in adjoining *table* 12.28. Proceeding in the usual way, the set of assignments made are shown in *table* 12.28.

Table 12.28

Machine \ Job	J_1	J_2	J_3	J_4	J_5
M_1	2	$\boxed{0}$	5	2	4
M_2	1	4	$\boxed{0}$	2	$\cancel{0}$
M_3	3	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
M_4	4	$\cancel{0}$	4	5	$\boxed{0}$
M_5	$\boxed{0}$	2	4	$\cancel{0}$	5

The optimum solution is :

Assign Job	To Machine	Cost (Rs.)
M_1	J_2	5
M_2	J_3	7
M_3	J_4	6
M_4	J_5	5
M_5	J_1	4
	Minimum total cost	= Rs. 27

Example 7. *ABC company is engaged in manufacturing 5 brands of packed snaks. It is having five manufacturing setups, each capable of manufacturing any of its brands, one at a time. The cost to make a brand on these setups vary according to the following table :*

	S_1	S_2	S_3	S_4	S_5
B_1	4	6	7	5	11
B_2	7	3	6	9	5
B_3	8	5	4	6	9
B_4	9	12	7	11	10
B_5	7	5	9	8	11

Assuming, five setups are $S_1, S_2, S_3, S_4,$ and S_5 *and five brands are* B_1, B_2, B_3, B_4 and B_5. *Find the optimum assignment of products on these setups resulting in the minimum cost.* **(C.A. Nov., 98)**

Solution. ***Step 1*** : Select the minimum element in each row and substract this element from every element in the row to get *table* 12.29.

Table 12.29

	S_1	S_2	S_3	S_4	S_5
B_1	0	2	3	1	7
B_2	4	0	3	6	2
B_3	4	1	0	2	5
B_4	2	5	0	4	3
B_5	2	0	4	3	6

Step 2 : Select the minimum element in each column and subtract this element from every element in the column to get the *table* 12.30.

Table 12.30

	S_1	S_2	S_3	S_4	S_5
B_1	0	2	3	0	5
B_2	4	0	3	5	0
B_3	4	1	0	1	3
B_4	2	5	0	3	1
B_5	2	0	4	2	4

Step 3 : We can attempt to make a complete set of assignments using only a single zero element in each row or each column. Since row B_5 contains a single zero, thus the assignment is ($B_5 \rightarrow S_2$). The other zero appearing in column S_2 is crossed out. Similarly the other assignments are made. Only four assignments can be made at this stage.

Table 12.31

	S_1	S_2	S_3	S_4	S_5
B_1	$\boxed{0}$	2	3	$\cancel{0}$	5
B_2	4	$\cancel{0}$	3	5	$\boxed{0}$
B_3	4	1	$\cancel{0}$	1	3
B_4	2	5	$\boxed{0}$	3	1
B_5	2	$\boxed{0}$	4	2	4

Step 4 : Since only four assignments could be made hence one more zero element need to be created by drawing the minimum number of horizontal and vertical lines. Mark the row B_3 which has no assignment. Mark column S_3 which has a zero in the marked row B_3. Mark row B_4 which has assignment in marked column S_3. Draw the lines through unmarked rows and marked columns to get *table* 12.32.

Table 12.32

	S_1	S_2	S_3	S_4	S_5	
B_1	$\boxed{0}$	2	3	0	5	
B_2	4	0	3	5	$\boxed{0}$	
B_3	4	1	$\cancel{0}$	1	3	✓
B_4	2	5	$\boxed{0}$	3	1	✓
B_5	2	$\boxed{0}$	4	2	4	
			✓			

Step 5 : To get number of lines drawn (*i.e.*, 4) equal to number of assignments to be made (*i.e.*, 5), we still need one more line. To create one more zero, the elements not covered by these lines are examined and smallest among them is selected which is '1'. Subtract this smallest element from every uncovered elc nents and add it to the element where the horizontal and vertical lines intersect to get the *Table* 12.33.

Table 12.33

	S_1	S_2	S_3	S_4	S_5
B_1	$\boxed{0}$	2	4	0	5
B_2	4	0	4	5	$\boxed{0}$
B_3	3	0	0	$\boxed{0}$	2
B_4	1	4	$\boxed{0}$	2	0
B_5	2	$\boxed{0}$	5	2	4

The optimum assignment is :

Assign Brand	To set up	Cost (Rs.)
B_1	S_1	4
B_2	S_5	5
B_3	S_4	6
B_4	S_3	7
B_5	S_2	5

Minimum total cost = Rs. 27

Alternative solution after Step 2 :

Step 3 : Draw the minimum number of lines to cover all zeros as shown in adjoining *table* 12.34.

Table 12.34

	S_1	S_2	S_3	S_4	S_5
B_1	0	2	3	0	5
B_2	4	0	3	5	0
B_3	4	1	0	1	3
B_4	2	5	0	3	1
B_5	2	0	4	2	4

Step 4 : Since the minimum number of lines to cover all zeros is 4 which is one less than the order of the matrix (= 5), the above table will not give the optimum solution. Subtract the minimum uncovered element (= 1) from all uncovered elements and add it to the elements lying at the intersection of two lines to get *table* 12.35.

Table 12.35

	S_1	S_2	S_3	S_4	S_5
B_1	0	3	4	0	5
B_2	4	1	4	5	0
B_3	3	1	0	0	2
B_4	1	5	0	2	0
B_5	1	0	4	1	3

Step 5 : The minimum number of lines to cover all zeros is 5 which is equal to the order of the matrix. The above table will give the optimum assignment as shown in *table* 12.35.

Example 8. *Five men are available to do five different jobs. From past records, the time (in hours) that each man takes to do each job is known and given in the following table :*

		Job				
		I	II	III	IV	V
	A	2	9	2	7	1
	B	6	8	7	6	1
Man	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

Find the assignment of men to jobs that will minimize the total time taken.

(A.I.M.A. (P.G. Dip. in Management) Dec. 95)

Solution. *Step 1* : Subtracting the smallest element of each row from every element of the corresponding row, we get the adjoining reduced matrix (*table* 12.36).

Table 12.36

Man \ Job	I	II	III	IV	V
A	1	8	1	6	0
B	5	7	6	5	0
C	3	5	4	2	0
D	3	1	6	2	0
E	4	2	8	4	0

Step 2 : Subtract the smallest element of each column from every element of the corresponding column to get the adjoining reduced matrix (*table* 12.37).

Table 12.37

Man \ Job	I	II	III	IV	V
A	0	7	0	4	0
B	4	6	5	3	0
C	2	4	3	0	0
D	2	0	5	0	0
E	3	1	7	2	0

Step 3 : Row 2 has a single zero in column 5. Make an assignment by putting square (☐) around it, and delete the other zeros in column 5 by marking '×'.

Now, column 4 has a single zero in row 3. We make an assignment by putting (☐) and cross the other zeros which is not yet crossed. Column 2 has a single zero in row 4, we make an assignment.

Table 12.38

Man \ Job	I	II	III	IV	V
A	$\boxed{0}$	7	$\cancel{0}$	4	1
B	3	5	4	2	$\boxed{0}$
C	2	4	3	$\boxed{0}$	1
D	2	$\boxed{0}$	5	$\cancel{0}$	1
E	2	$\cancel{0}$	6	1	$\cancel{0}$

It may be noted that there are no remaining zeros, and row *E* and column *III* has no assignment. Thus, the optimum solution is not reached at this stage and we proceed to the following important steps.

Step 4 : Draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. The following systematic procedure may help to draw the minimum set of lines :

(i) For simplicity, first make the table 12.39, again.

Table 12.39

Man \ Job	I	II	III	IV	V	
A	$\boxed{0}$	7	~~0~~	4	~~0~~	
B	4	6	5	3	$\boxed{0}$	✓
C	2	4	3	$\boxed{0}$	~~0~~	
D	2	$\boxed{0}$	5	~~0~~	~~0~~	✓
E	3	1	7	2	~~0~~	
					✓	

(ii) Secondly, mark (√) row 5 in which there is no assignment, *i.e.*, the last row.

(iii) Then mark (√) column 5, which has a zero in the marked row.

(iv) Next mark (√) row 2, which has assignment in the marked column.

(v) Draw the minimum number of lines covering the unmarked rows and the marked columns.

Step 5 : Examine the elements that do not have a line through them. Select the smallest of these and subtract it from all the elements that do not have a line through them. Add this element to every element lying at the intersection of two lines. Leave the remaining elements of the matrix unchanged.

Table 12.40

Man \ Job	I	II	III	IV	V
A	$\boxed{0}$	7	~~0~~	4	1
B	4	5	4	2	$\boxed{0}$
C	2	4	3	$\boxed{0}$	1
D	2	0	5	~~0~~	1
E	2	~~0~~	6	1	~~0~~

Step 6 : Repeat the steps to obtain optimum solution.

Table 12.41

Man \ Job	I	II	III	IV	V
A	~~0~~	9	$\boxed{0}$	6	3
B	1	5	2	2	$\boxed{0}$
C	$\boxed{0}$	4	1	~~0~~	1
D	~~0~~	~~0~~	3	$\boxed{0}$	1
E	~~0~~	$\boxed{0}$	4	1	~~0~~

Thus in table 12.41, there are no remaining zeros, and every row and column has assignment, optimum solution is reached.

Hence the minimum time taken $= 2 + 1 + 4 + 3 + 3 = 13$ hours.

Example 9. *An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination. This is given in the matrix form in adjacent table :*

Man \ Job	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Find the optimum assignment that will result in minimum manhours needed. **(Madras (M. Com.) 98)**

Solution. ***Steps 1.*** Subtracting the smallest element of each row from all the elements of that row and then in the second matrix subtracting the smallest element of each column from all the elements of that column, the initial feasible solution determined by zeros is obtained.

Table 12.42

Man \ Job	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	1	0
4	0	2	2	0

Steps 2. We now examine each row successively for a single zero. Enrectangling these zeros and crossing (×) all the remaining zeros in the respective columns and then repeating same procedure for each column, the adjacent *table* 12.43 is obtained.

Table 12.43

Man \ Job	A	B	C	D
1	3	1	$\boxed{0}$	3
2	5	7	$\cancel{0}$	1
3	2	$\boxed{0}$	1	$\cancel{0}$
4	$\boxed{0}$	2	2	$\cancel{0}$

But still optimum assignment is not reached, since no zero has been marked in the *second* row and *fourth* column.

Steps 3. We now draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. This has been obtained in *Table* 12.44. Select the smallest element not covered by the lines (*i.e.*, 1) and subtract it from all the elements not covered by the lines and add the same to the elements at the intersection of the lines. We thus obtain *Table* 12.45 providing the second feasible solution to the problem.

Table 12.44

Man \ Job	A	B	C	D	
1	3	1	$\boxed{0}$	3	
2	5	7	$\cancel{0}$	1	
3	2	$\boxed{0}$	1	$\cancel{0}$	L_2
4	$\boxed{0}$	2	2	$\cancel{0}$	L_3
			L_1		

Table 12.45

Man \ Job	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

Step 4. Repeating step 2, we make the 'zero-assignments' as shown in the table 12.46 a. It may be noted that an assignment problem can have more than one optimum solution. The other solution is shown in table 12.46 b.

Table 12.46 a

Man \ Job	A	B	C	D
1	2	$\boxed{0}$	$\cancel{0}$	2
2	4	6	$\boxed{0}$	$\cancel{0}$
3	2	$\cancel{0}$	2	$\boxed{0}$
4	$\boxed{0}$	2	3	$\cancel{0}$

Optimum solution I

Man	Job	Man hours
1	B	3
2	C	2
3	D	7
4	A	5

Table 12.46 b

Man \ Job	A	B	C	D
1	2	$\cancel{0}$	$\boxed{0}$	2
2	4	6	$\cancel{0}$	$\boxed{0}$
3	2	$\boxed{0}$	2	$\cancel{0}$
4	$\boxed{0}$	2	3	$\cancel{0}$

Optimum solution II

Man	Job	Man hours
1	C	2
2	D	6
3	B	4
4	A	5

Example 10. *A private firm employs typists on hourly piece rate basis for their daily work. Five typists are working in that firm and their charges and speeds are different. On the basis of some earlier understanding, only one job is given to one typist and the typist is paid for full hours even when he or she works for a fraction of an hour. Find the least cost allocation for the following data :*

Typist	Rate per hour (Rs.)	Number of pages typed prr hr.
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	No. of pages
P	199
Q	175
R	145
S	298
T	178

[C.A., (Nov.) 96; Delhi (M.B.A.) Nov. 96]

Solution.

Step 1. The following matrix gives the cost incurred if the i th typist $(i = A, B, C, D, E)$ executes the j th job $(j = P, Q, R, S, T)$.

Table 12.47

Typist \ Job	*P*	*Q*	*R*	*S*	*T*
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

Steps 2. Subtracting the minimum element of each row from all its elements in turn, the adjoining matrix reduces to :

Table 12.48

Typist \ Job	*P*	*Q*	*R*	*S*	*T*
A	20	10	0	60	10
B	24	12	0	66	12
C	18	9	0	57	12
D	20	12	0	60	12
E	20	8	0	56	12

Now subtract the minimum element of each column from all its elements, in turn, the above matrix reduces to

Table 12.49

Typist \ Job	*P*	*Q*	*R*	*S*	*T*
A	2	2	0	4	0
B	6	4	0	10	2
C	0	1	0	1	2
D	2	4	0	4	2
E	2	0	0	0	2

Step 3. Since there are only 4 lines (< 5) to cover all zeros, optimum assignment cannot be made. The minimum uncovered element is 2. We, therefore subtract the minimum uncovered element '2' from all uncovered elements, add this value to all junction values and leave the other elements undisturbed, as shown in the adjoining matrix;

Table 12.50

Typist \ Job	*P*	*Q*	*R*	*S*	*T*
A	2	2	2	4	0
B	4	2	0	8	0
C	0	1	2	1	2
D	0	2	0	2	0
E	2	0	2	0	2

Step 4. Since the minimum number of lines required to cover all the zeros is only 4 (< 5), optimum assignment cannot be made at this stage also. The minimum uncovered element is 1. Repeating the usual procedure again, we get the adjoining matrix :

Table 12.51

Typist \ Job	*P*	*Q*	*R*	*S*	*T*
A	2	1	2	3	0
B	4	1	0	7	0
C	0	0	2	0	2
D	0	1	0	1	0
E	3	0	3	0	3

Step 5. Since the minimum number of lines to cover all zeros is equal to 5, this matrix will give optimum solution. The optimum assignment is made in the matrix given below :

Table 12.52

Typist \ Job	*P*	*Q*	*R*	*S*	*T*
A	2	1	2	3	$\boxed{0}$
B	4	1	$\boxed{0}$	7	$\cancel{0}$
C	$\cancel{0}$	$\boxed{0}$	2	$\cancel{0}$	2
D	$\boxed{0}$	1	$\cancel{0}$	1	$\cancel{0}$
E	3	$\cancel{0}$	3	$\boxed{0}$	3

Typist		Job		Cost (Rs.)
A	→	*T*	—	75
B	→	*R*	—	66
C	→	*Q*	—	66
D	→	*P*	—	80
E	→	*S*	—	112
			Total	Rs. 399

Remark. It may be noted that the above solution is not unique as a alternate optimum solution also exists.

EXAMINATION PROBLEMS

1. Solve the following assignment problems

(a)

Work \ Man	1	2	3	4
I	12	30	21	15
II	18	33	9	31
III	44	25	24	21
IV	23	30	28	14

(b)

Operators \ Jobs	I	II	III	IV
A	10	12	9	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

[Ans. (a) I → 1 , II → 3 , III → 2 , IV → 4 , min cost = 60 ; (b) A → III , B → I , C → II , D → IV , min cost = 37 .]

2. Solve the following cost-minimizing problems :

(a)

Jobs	I	II	III	IV	V
A	45	30	65	40	55
B	50	30	25	60	30
C	25	20	15	20	40
D	35	25	30	30	20
E	80	60	60	70	50

(b)

Machines	I	II	III	IV	V
A	2	9	2	7	1
B	6	8	7	6	1
C	4	6	5	3	1
D	4	2	7	3	1
E	5	3	9	5	1

[Delhi B.Sc. (Math.) 90]

(c)

Machines \ Jobs	I	II	III	IV	V
A	11	10	18	5	9
B	14	13	12	19	6
C	5	3	4	2	4
D	15	18	17	9	12
E	10	11	19	6	14

[Delhi B.Sc. (Maths) 93, 89]

[Ans. (a) (A → III , B → V , C → I , D → IV , E → II) or (A → III , B → V , C → IV , D → I , E → II) or (A → III , B → V , C → IV , D → II , E → I) , min cost = 13 .
(b) (A → II, B → III; C → I, D → IV, E → V) or (A → II, B → III, C → IV, D → I, E → V), min cost = 160
(c) A → II , B → V , C → III , D → IV , E → I , and min cost = 39]

3. A team of 5 horses and 5 riders has entered a jumping show contest. The number of penalty points to be expected when each rider rides any horse is shown below :

Horses	Riders				
	R_1	R_2	R_3	R_4	R_5
H_1	5	3	4	7	1
H_2	2	3	7	6	5
H_3	4	1	5	2	4
H_4	6	8	1	2	3
H_5	4	2	5	7	1

How should the horses be alloted to the riders so as to minimize the expected loss of the team.
[Ans. $H_1 \to R_5$, $H_2 \to R_1$, $H_3 \to R_4$, $H_4 \to R_3$, $H_5 \to R_2$; min. loss = 8 .]

4. A company has six jobs to be done on six machines; any job can be done on any machine. The time in hours taken by the machines for the different jobs are as given below. Assign the machines to jobs so as to minimize the total machine hours.

Machines	Jobs					
	1	2	3	4	5	6
1	2	6	7	3	8	7
2	6	1	3	9	7	3
3	3	6	5	7	3	5
4	2	2	7	8	4	8
5	4	9	6	8	7	6
6	7	5	5	7	7	5

[Delhi B.Sc. (Math.) 91]

[Ans. 1 → 4 , 2 → 2 , 3 → 5 , 4 → 1 , 5 → 3 , 6 → 6 or 1 → 4 , 2 → 2 , 3 → 5 , 4 → 1 , 5 → 6 , 6 → 3 or 1 → 4 , 2 → 2 , 3 → 5 , 4 → 2 , 5 → 1 , 6 → 6 ; min time = 20 hrs .]

5. A small aeroplane company, operating seven days a week serves three cities A, B, and C according to the schedule shown in the following table.The layover cost per stop is roughly proprotional to the square of the layover time. How should planes be assigned to the flights so as to minimize the total layover cost ?

Flight No. and Index	From	Departure	To	Arrival
A_1B	A	09 *AM*	B	*Noon*
A_2B	A	10 *AM*	B	01 PM
A_3B	A	03 *PM*	B	06 PM
A_4C	A	08 *PM*	C	Mid. Night
A_5C	A	10 *PM*	C	02 AM
B_1A	B	04 *AM*	A	07 AM
B_2A	B	11 *AM*	A	02 PM
B_3A	B	03 *PM*	A	06 PM
C_1A	C	07 *AM*	A	11 AM
C_2A	C	03 *PM*	A	07 PM

[Agra 98]

[Ans.					
Flight No :	1	2	3	4	5
Departure Route :	A_1B	A_2B	A_3B	C_1A	C_2A
Arrival Route :	B_3A	B_1A	B_2A	A_4C	A_5C]

6. A trip from Madurai to Trivandrum takes 6 hours by bus. A typical time-table of bus services in both directions is given below :

Madurai-Trivendrum			Trivandrum-Madurai		
Route No.	Depart	Arrive	Route No	Depart	Arrive
a	06·00	12·00	1	05·30	11·30
b	07·30	13·30	2	09·00	15·00
c	11·30	17·30	3	15·00	21·00
d	19·00	01·00	4	18·30	00·30
e	00·30	06·30	5	00·00	06·00

The cost of providing this service by the transport company depends upon the time spent by the bus crew (driver and conductor) away from their places in addition to service times. There are five crews. There is a constraint that every crew should be provided with 4 hours of rest before return trip again and should not wait for more them 24 hours for the return trip. The company has residential facilities for the crew at Madurai as well as at Trivendrum. Obtain the pairing of routes so as to minimize the cost. **[VTU 2002; Madurai 93]**

[Ans.					
Crew :	1	2	3	4	5
Residence :	Madurai	Trivendrum	Trivendrum	Madurai	Trivendrum
Service No. :	d1	2e	3a	b4	5c
Waiting Time (hrs) :	4·5	9·5	9·0	5·0	5·5

Minimum total waiting time = 33·30 hours.]

12.6. UNBALANCED ASSIGNMENT PROBLEM

If the cost matrix of an assignment problem is not a square matrix (number of sources is not equal to the number of destinations), the assignment problem is called as *Unbalanced Assignment Problem.* In such cases, *fictitious* rows and/or columns with zero costs are added in the matrix so as to form a square matrix. Then the usual assignment algorithm can be applied to this resulting balanced problem.

Q. What is an unbalanced assignment problem ? Explain the various steps involved in solving it. **[JNTU (B. Tech.) 2003; IGNOU 2001, 99, 97, 96]**

Example 11. *A company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated as follows (in hundreds of rupees) :*

Table 12.53

		Jobs				
		1	2	3	4	5
Machines	1	2.5	5.0	1.0	6	1.0
	2	2.0	5.0	1.5	7	3.0
	3	3.0	6.5	2.0	8	3.0
	4	3.5	7.0	2.0	9	4.5
	5	4.0	7.0	3.0	9	6.0
	6	6.0	9.0	5.0	10	6.0

Solve the problem assuming that the objective is to minimize the total cost.

Solution. Introducing one more column for a fictitious job (say, job 6) in the cost matrix in order to get the following balanced assignment problem. The cost corresponding to sixth column are always taken as zero.

Table 12·54

		Jobs					
		1	2	3	4	5	6 (Dummy)
Machines	1	2·5	5·0	1·0	6	1·0	0
	2	2·0	5·0	1·5	7	3·0	0
	3	3·0	6·5	2·0	8	3·0	0
	4	3·5	7·0	2·0	9	4·5	0
	5	4·0	7·0	3·0	9	6·0	0
	6	6·0	9·0	5·0	10	6·0	0

Since the problem can be solved as in usual practice, there is no need to give the detailed solution here, because it has already been explained earlier.

Example 12. *A Methods Engineer wants to assign four new methods to three work centres. The assignment of the new methods will increase production and they are given below. If only one method can be assigned to a work centre, determine the optimum assignment:*

		Increase in production (unit)		
		Work centres		
		A	B	C
Methods	1	10	7	8
	2	8	9	7
	3	7	12	6
	4	10	10	8

Solution. The given problem is of maximization type, since the elements of the given matrix relate to increase in production of units due to introduction of new methods. First of all, convert it into minimization problem by subtracting each element of the given matrix from maximum element 12. Since the problem is unbalanced one, introduce a dummy work centre.

Table 12.55

Method \ Work centre	A	B	C	Dummy
1	2	5	4	0
2	4	3	5	0
3	5	0	6	0
4	2	2	4	0

Subtracting the smallest element of each column from all elements of that column, we get the adjoining *table* 12.56.

Table 12.56

Method \ Work centre	A	B	C	Dummy
1	0	5	0	0
2	2	3	1	0
3	3	0	2	0
4	0	2	0	0

Since the minimum number of horizontal and vertical lines to cover up all zeros is 4, the reduced matrix will give the optimum solution.

Table 12.57

Method \ Work centre	A	B	C	Dummy
1	$\boxed{0}$	5	0	0
2	2	3	1	$\boxed{0}$
3	3	$\boxed{0}$	2	0
4	0	2	$\boxed{0}$	0

The allocations as obtained from the above process are $1 \to A$, $2 \to$ Dummy, $3 \to B$, $4 \to C$. The total production under the above assignment is :

10 units + 12 units + 8 unit = 30 units.

Example 13. *A city corporation has decided to carry out road repairs on main four arteries of the city. The government has agreed to make a special grant of Rs. 50 lakhs towards the cost with a condition that the* *repairs must be done at the lowest cost and quickest time. If conditions warrant, then a supplementary token grant will also be considered favourably. The corporation has floated tenders and 5 contractors have sent in their bids. In order to expedite work, one road will be awarded to only one contractor.*

Cost of repairs (Rs. lakhs)

		R_1	R_2	R_3	R_4
	C_1	9	14	19	15
	C_2	7	17	20	19
Contractors/Road	C_3	9	18	21	18
	C_4	10	12	18	19
	C_5	10	15	21	16

(i) Find the best way of assigning the repair work to the contractors and the costs.

(ii) If it is necessary to seek supplementary grants, then what should be amount sought ?

(iii) Which of the five contractors will be unsuccessful in his bid ?

Solution. Since this is an assignment problem with 5 contractors and 4 roads, a dummy road 'R_5' with zero cost of repairing for each contractor is introduced to make the problem balanced.

Step 1 (a) Row subtraction

Road / Contractors	R_1	R_2	R_3	R_4	R_5
C_1	9	14	19	15	0
C_2	7	17	20	19	0
C_3	9	18	21	18	0
C_4	10	12	18	19	0
C_5	10	15	21	16	0

Step 1 (b) Column subtraction

Road / Contractors	R_1	R_2	R_3	R_4	R_5
C_1	2	2	1	0	0
C_2	0	5	2	4	0
C_3	2	6	3	3	0
C_4	3	0	0	4	0
C_5	3	3	3	1	0

Step 2. Draw minimum straight lines to cover all zeros.

2	2	1	0	0
0	5	2	4	0
2	6	3	3	0
3	0	0	4	0
3	3	3	1	0

Step 3. Smallest uncovered number is then subtracted from uncovered numbers added to numbers at intersection of two lines.

2	2	1	0	1
0	5	2	4	1
1	5	2	2	0
3	0	0	4	1
2	2	2	0	0

Step 4. Return to *step* 2. cover all zeros, since the number of lines is 4, the optimality criteria is not satisfied.

1	1	0	0	1
0	5	2	5	2
0	4	1	2	0
3	0	0	5	2
1	1	1	0	0

Step 5. Return to *step* 3. All rows and columns have single allocation and hence optimality criteria is satisfied.

1	1	$\boxed{0}$	$\cancel{0}$	1
$\boxed{0}$	5	2	5	2
$\cancel{0}$	4	1	2	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	5	2
1	1	1	$\boxed{0}$	$\cancel{0}$

Thus allotments are as follows

(i) $R_1 \xrightarrow{7} C_2$, $R_2 \xrightarrow{12} C_4$, $R_3 \xrightarrow{19} C_1$, $R_4 \xrightarrow{16} C_5$, $R_5 \xrightarrow{0} C_3$, Total cost = Rs. 54 lacs

(ii) Since cost exceeds 50 lacs, the excess amount of Rs. 4 (54 – 50) lacs is to be sought as supplimentary grant.

(iii) Contractor C_3 who has been assigned the dummy row (Road R_5) loses out in the bid.

EXAMINATION PROBLEMES

Q. 1. A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table :

	Machine W	X	Y	Z
Job A	18	24	28	32
Job B	8	13	17	19
Job C	10	15	19	22

What are the job assignments which will minimize the cost?

[JNTU (MCA III) 2004, (B. Tech.) 2003; Gauhati (M.C.A.) 92]

[Ans. A → W , B → X , C → Y ; or A → W , B → Y , C → X ; min cost = 50]

2. In a machine shop, a supervisor wishes to assign five jobs among six machines. Any one of the jobs can be processed completely by any one of the machines as given below :

	Machine A	B	C	D	E	F
Job 1	13	13	16	23	19	9
2	11	19	26	16	17	18
3	12	11	4	9	6	10
4	7	15	9	14	14	13
5	9	13	12	8	14	11

The assignment of jobs to machines be on a one-to-one basis. Assign the jobs to machines so that the total cost is minimum. Find the minimum total cost. **[I.A.S. (Maths.) 98]**

[Ans. 1 → F, 2 → A, 3 → E, 4 → C. 5 → D, min cost = 43 .]

3. A department head has six jobs and five subordinates. The subordinates differ in their efficiency and the tasks differ in their intrinsic difficulty. The department head estimates the time each man would take to perform each task as given in the effectiveness matrix below :

	Task A	B	C	D	E	F
Man 1	20	15	26	40	32	12
2	15	32	46	26	28	20
3	11	15	2	12	6	14
4	8	24	12	22	22	20
5	12	20	18	10	22	15

Only one task can be assigned to one man. Determine how should the jobs be allocated so as to minimize the total man hours. Find the minimum total man hours.

[Ans.. 1 → F, 2 → A, 3 → E, 4 → C, 5 → D ; or 1 → B , 2 → F , 3 → C, 4 → A, 5 → D ; min time = 55 hrs .]

4. A truck company on a particular day has 5 trucks for sending material to 6 terminals. The cost of sending material from some destination to different trucks will be different as given by the cost matrix below. Find the assignment of 4 trucks to 4 terminals out of six at the minimum cost.

	Trucks A	B	C	D	E
Terminals 1	3	6	2	6	5
2	7	1	4	4	7
3	3	8	5	8	3
4	6	4	3	7	4
5	5	2	4	3	2
6	5	7	6	2	5

[Ans. 1 → C, 2 → B, 3 → A, 6 → D ; min cost = 8]

5. Solve the following unbalanced assignment problem of minimizing total time for doing all the jobs :

		Job 1	2	3	4	5
	1	6	2	5	2	6
	2	2	5	8	7	7
Operator	3	7	8	6	9	8
	4	6	2	3	4	5
	5	9	3	8	9	7
	6	4	7	4	6	8

[Ans. 1 → 4, 2 → 1, 3 → dummy 6, 4–5, 5–2, 6 → 3, min. time = 16 units]

6. To stimulate interest and provide an atmosphere for intellectual discussion a finance faculty in a management school decides to hold special seminars on four contemporary topics–leasing, portfolio management, private mutual funds, swaps and options. Such seminars should be hold once per week in the afternoons. However, scheduling these seminars (one for each topic, and not more than one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows :

	Leasing	Portfolio management	Private Mutual funds	Swaps and options
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimal schedule of the seminars. Also find-out the total number of students who will be missing at least one seminar. **[C.A. (Nov.) 92]**

[Ans. Optimal Schedule	**:**	**No. of Students Missing**
Monday	: Swaps and options	20
Tuesday	: No. seminar	0
Wednesday	: Partfolio management	20
Thursday	: Private mutual funds	20
Friday	: Leasing	10
	Total	70

7. A hospital wants ot purchase three different types of medical equipment and five manufacturers have come forward to supply one or all the three machines. However, the hospitals policy is not to accept more than one machine from any one of the manufacturers. The data relating to the price (in thousand of rupees) quoted by the different manufacturers are given below :

		Machines 1	2	3
	A	30	31	27
	B	28	29	26
Manufacturers	C	29	30	28
	D	28	31	27
	E	31	29	26

Determine how best the hospital can purchase three machines. **(Delhi (MBA) 2001)**

12.7. VARIATIONS IN THE ASSIGNMENT PROBLEM

In this section, we shall discuss two variations of the assignment problem.

12.7-1 The Maximal Assignment Problem

Sometimes, the assignment problem deals with the maximization of an objective function rather than to minimize it. For example, it may be required to assign persons to jobs in such a way that the expected profit is maximum. Such problem may be solved easily by first converting it to a minimization problem and then applying the usual procedure of assignment algorithm. This conversion can be very easily done by subtracting from the highest element, all the elements of the given profit matrix; or equivalently, by placing minus-sign before each element of the profit-matrix in order to make it cost-matrix.

Q. 1. When in an assignment problem, the objective function is of maximization instead of minimization, what modifications are needed in the assignment algorithm to achieve this maximal assignment ?

2. What other variations of an assignment problem are possible ?

3. How can you maximize an objective function in the assignment problem.

Following examples will make the procedure clear.

Example 14. ***(Maximization Problem).*** *A company has 5 jobs to be done. The following matrix shows the return in rupees on assigning ith* ($i = 1, 2, 3, 4, 5$) *machine to the jth job* (j = A, B, C, D, E). *Assign the five jobs to the five machines so as to maximize the total expected profit.*

		Jobs				
		A	B	C	D	E
	1	5	11	10	12	4
	2	2	4	6	3	5
Machines	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

[Kerala B.Sc. (Math.) 90]

Solution. ***Step 1. Converting from Maximization to Minimization :***

Since the highest element is 14, so subtracting all the elements from 14, the following reduced cost (opportunity loss of maximum profit) matrix is obtained.

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	2

Step 2. Now following the usual procedure of solving an assignment problem, an optimal assignment is obtained in the following table :

1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	5
$\cancel{0}$	13	$\cancel{0}$	5	$\boxed{0}$
5	1	7	$\boxed{0}$	5
3	$\boxed{0}$	9	4	5
$\boxed{0}$	3	3	1	5

This table gives the optimum assignment as : $1 \to C$, $2 \to E$, $3 \to D$, $4 \to B$, $5 \to A$; with maximum profit of Rs. 50.

Example 15. ***(Maximization Problem).*** *A company has four territories open, and four salesmen available for assignment. The territories are not equally rich in their sales potential ; it is estimated that a typical salesman operating in each territory would bring in the following annual sales :*

Territory	: I	II	III	IV
Annual sales (Rs.)	: 60,000	50,000	40,000	30,000

Four salesmen are also considered to differ in their ability : it is estimated that, working under the same conditions, their yearly sales would be proportionately as follows :

Salesman	: *A*	*B*	*C*	*D*
Proportion	: 7	5	5	4

If the criterion is maximum expected total sales, then intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on. Verify this answer by the assignment technique. **[JNTU (B. Tech.) 2003; VTU Aug. 2002; Delhi (MBA) 2001]**

Solution. ***Step 1. To construct the effectiveness matrix :***

In order to avoid the fractional values of annnual sales of each salesman in each territory, it will be rather convenient to consider the sales for 21 years (the sum of proportions : $7+5+5+4=21$), taking Rs. 10,000 as one unit. Divide the individual sales in each territory by 21, if the annual sales by salesman are required.

Thus, the sales matrix for maximization is obtained as follows :

Table 12·58
Sales in 10 thousand of rupees

Sales proportion ↓	6 I	5 II	4 III	3 IV
7 A	42	35	28	21
5 B	30	25	20	15
5 C	30	25	20	15
4 D	24	20	16	12

Step 2. ***(To convert 'the maximum sales matrix' to 'minimum sales matrix'.)***

The problem of 'maximization' can be converted to 'minimization' one, by simply multiplying each ment of given matrix (*Table 12·58*) by –1. Thus resulting matrix becomes :

Table 12·59

	I	II	III	IV
A	– 42	– 35	– 28	– 21
B	– 30	– 25	– 20	– 15
C	– 30	– 25	– 20	– 15
D	– 24	– 20	– 16	– 12

Step 3. Subtracting the smallest element in each row from every element in that row, we get the reduced matrix (*Table 12·60*).

Table 12·60

0	7	14	21
0	5	10	15
0	5	10	15
0	4	8	12

Step 4. Subtract the smallest element in each column from every element in that column to get the second reduced matrix (*Table 12·61*)

Table 12·61

L_2			
0	3	6	9
0	1	2	3
0	1	2	3
0	0	0	0 ▶ L_1

Since all zeros in *Table 12·61* can be covered by minimum number of lines (L_1, L_2), which is less than 4 (the number of rows in the matrix), the optimal assignment is not possible at this stage.

Step 5. In *Table 12·61*, select the minimum element '1' among all uncovered elements. Then subtract this value 1 from each uncovered element, and add 1 at the intersection of two lines L_1, L_2. Thus, the revised matrix is obtained as *Table 12·62*.

Table 12.62

L_2	L_3			
0	2	5	8	
0	0	1	2	
0	0	1	2	
0	0	0	0	L_1

Step 6. Again, repeat *Step 5.* Since the minimum number of lines (L_1 , L_2 , L_3) in *Table 12·62* to cover all zeros is less than 4 (the number of rows/columns), subtract the min. element 1 from all uncovered elements and add 1 at the intersection of lines (L_1 , L_2) and (L_1 , L_3) . Then find the optimal assignment as explained in *Step 7.*

Step 7. ***To find an optimal assignment.***

Since there is a single zero element in row 1 and column 4 only, make the zero assignment by putting '□' around these two zeros and cross-out other zeros in column 1 and row 4. Other zero-assignments are quite obvious from the following tables :

Table 12·63 (a)

	I	II	III	IV
A	$\boxed{0}$	2	4	7
B	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
C	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
D	2	1	$\cancel{0}$	$\boxed{0}$

Table 12·63 (b)

	I	II	III	IV
A	$\boxed{0}$	2	4	
B	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	
C	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
D	2	1	$\cancel{0}$	$\boxed{0}$

Thus, two possible solutions are : *(i)* A-I, B-II, C-III, D-IV ; *(ii)* A-I, B-III, C-II, D-IV.

Both the solutions show that the best salesman *A* is assigned to the richest territory I, the worst salesman *D* to the poorest territory IV. Salesman *B* and *C* being equally good, so they may be assigned to either II or III. This verifies the answer.

Example 16. *A manufacturing company has four zones A, B, C, D and four sales engineers P, Q, R, S respectively for assignment. Since the zones are not equally rich in sales potential, it is estimated that a particular engineer operating in a particular zone will bring the following sales :*

Zone A	:	4,20,000
Zone B	:	3,36,000
Zone C	:	2,94,000
Zone D	:	4,62,000

The engineers are having different sales ability. Working under the same conditions, their yearly sales are proportional to 14, 9, 11 and 8 respectivley. The criteria of maximum expected total sales is to be met by assigning the best engineer to the richest zone, the next best to the second richest zone and so on.

Find the optimum assignment and the maximum sales. **(C.A. (May) 98)**

Solution. ***Step 1*** **: Construct the Effectiveness Matrix.** To avoid the fractional values of annual sales of each sales engineer in each zone, for convenience consider their yearly sales as 42 (*i.e.*, the sum of sales proportions), taking Rs. 1000 as one unit. Now divide the individual sales in each zone by 42 to obtain the required annual sales by each sales-engineer. The maximum sales matrix so obtained is given in *table* 12.64

Table 12.64 : EFFECTIVENESS MATRIX

		Zones				Sales proportion
		A	B	C	D	
Sales Engineer	P	140	112	98	154	14
	Q	90	72	63	99	9
	R	110	88	77	121	11
	S	80	64	56	88	8
Sales (in Rs. 1000)		10	8	7	11	

***Step 2* : Converting Maximization problem into Minimization Problem** : The given maximization assignment problem (*table* 12.64) can be converted into a minimization assignment problem by subtracting from the highest element (*i.e.*, 154), all the elements of the given table. The resulting matrix so obtained is given in *table* 12.65 below :

Step 3 : Subtracting the smallest element in each row from every element in that row and smallest element in each column from every element in that column, we get the following reduced matrix (*table* 12.66). Since all zeros in table 12.65 can be covered by minimum number of lines (L_1, L_2) which is less than the number of rows (4) in the matrix, the optimum assignment is not possible at this stage and we pass to the next step.

Table 12.65 : EQUIVALENT COST TABLE

Sales Engr. \ Zone	A	B	C	D
P	14	42	56	0
Q	64	82	91	55
R	44	66	77	33
S	74	90	98	66

Table 12.66

Sales Engr. \ Zone	A	B	C	D	
P	6	18	24	0	
Q	1	3	4	0	
R	3	9	12	0	
S	0	0	0	0	L_1
				L_2	

Table 12.67

Sales Engr. \ Zone	A	B	C	D	
P	5	17	23	0	
Q	0	2	3	0	
R	2	8	11	0	
S	0	0	0	0	L_1
	L_3			L_2	

Step 4. Select the minimum element '1' among all uncovered elements and subtract this value from each uncovered element and add '1' at the intersection of two lines L_1, L_2. Draw more minimum possible number of lines so as to cover the new zeros (*table* 12.67).

Step 5. Since the number of lines is still less than the order of the sale matrix, we repeat the procedure and obtain *table* 12.68 and *table* 12.69. The table shows that the number of lines is equal to order of the sale matrix and hence an optimum solution is obtained.

Table 12.68

Zone → / Sales Engr. ↓	A	B	C	D	
P	5	15	21	0	
Q	0	0	1	0	L_2
R	2	6	9	0	
S	0	0	0	3	L_3
				L_1	

Table 12.69

Zone → / Sales Engr. ↓	A	B	C	D	
P	3	13	19	0	
Q	0	0	1	0	L_2
R	0	4	7	0	
S	3	0	0	3	L_3
	L_4			L_1	

Step 6. To determine the optimum assignment, we first observe that there is only single zero element in row 1 and column 3, so we make the zero assignment by putting '☐' around these two zeros and cross-out other zeros in column 4 and row 4. The remaining zero-assignments are quite obvious from the adjoining *table* 12.70.

Table 12.70

Zone → / Sales Engr. ↓	A	B	C	D
P	3	13	19	$\boxed{0}$
Q	~~0~~	$\boxed{0}$	1	2
R	$\boxed{0}$	4	7	~~0~~
S	2	~~0~~	$\boxed{0}$	5

The optimum solution is obtained and the assignment is :

$$P \to D, Q \to B, R \to A \text{ and } S \to C$$

The solution shows that the best salesman P is assigned to the richest zone D and the worse salesman S to the poorest zone C. The second best salesman to the next richest zone A and so on.

$$\text{Maximum sales} = \text{Rs. } 154 + 72 + 110 + 56 = \text{Rs. } 392 \text{ thousands}$$
$$= \text{Rs. } 392000.$$

Example 17. *Five salesmen are to be assigned to five territories. Based on the past performance, the following table shows the annual sales (in rupees lakhs) that can be generated by each salesman in each territory. Find the optimum assignment.*

Salesman	Territory T_1	T_2	T_3	T_4	T_5
S_1	26	14	10	12	9
S_2	31	27	30	14	16
S_3	15	18	16	25	30
S_4	17	12	21	30	25
S_5	20	19	25	16	10

[A.I.M.A. (P.G. (Dip. in Management)), Dec. 96]

Solution. *Step 1.* Since the matrix represents the sales which can be generated by each territory, the objective function of the assignment problem is, therefore, to maximize the total sales generated. But the algorithm for assignment problem is for minimization of the objective function, We, therefore, convert the given problem to minimization problem, by subtracting all the elements of the given matrix from the maximum element 31 to obtain the adjoining matrix.

Salesman \ Territory	T_1	T_2	T_3	T_4	T_5
S_1	5	17	21	19	22
S_2	0	4	1	17	15
S_3	16	13	15	6	1
S_4	14	19	10	1	6
S_5	11	12	6	15	21

Step 2. Row subtraction

Salesman \ Territory	T_1	T_2	T_3	T_4	T_5
S_1	0	12	16	14	17
S_2	0	4	1	17	15
S_3	15	12	14	5	0
S_4	13	18	9	0	5
S_5	5	6	0	9	15

Column subtraction

Salesman \ Territory	T_1	T_2	T_3	T_4	T_5
S_1	0	8	16	14	17
S_2	0	0	1	17	15
S_3	15	8	14	5	0
S_4	13	14	9	0	5
S_5	5	2	0	9	15

Step 3. Minimum straight lines to cover zeros.

Territory → / Salesman ↓	T_1	T_2	T_3	T_4	T_5	
S_1	0	12	16	14	17	
S_2	0	4	1	17	15	→ L_2
S_3	15	12	14	5	0	
S_4	13	18	9	0	5	
S_5	5	6	0	9	15	
	L_1		L_3	L_4	L_5	

Step 4. Since number of lines is 5, the optimality criteria is satisfied.

Territory → / Salesman ↓	T_1	T_2	T_3	T_4	T_5
S_1	$\boxed{0}$	8	16	14	17
S_2	$\cancel{0}$	$\boxed{0}$	1	17	15
S_3	15	8	14	5	$\boxed{0}$
S_4	13	14	9	$\boxed{0}$	5
S_5	5	2	$\boxed{0}$	9	15

The optimum assignment is :

$$S_1 \to T_1, S_2 \to T_2, S_3 \to T_5, S_4 \to T_4, S_5 \to T_3$$

and the maximum sales generated are :

$$26 + 27 + 30 + 30 + 25 = 138.$$

EXAMINATION PROBLEMS

1. In an assignment problem, there are 12 workers and 12 jobs to be done. Only one man can work on any one job. What is the total number of different possible ways of assignment if the jobs to the worker 3 ? **[IGNOU 99 (Dec.)]**
[Ans. 12 ! = 479001600 ways]

2. A marketing manager has 5 salesman and 5 sales districts. Considering the capabilities of the salesman and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows :

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment of salesman to districts that will result in a maximum sale.
[Agra 98; Rohil. 90; Bharathidasan B.Sc. 90]
[Ans. $1 \to B$, $2 \to A$, $3 \to E$, $4 \to C$ and $5 \to D$, max. profit = Rs. 191]

3. The owner of a small mechine shop has four machinists available to do jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows :

	1	2	3	4
A	32	41	57	18
B	48	54	62	34
C	20	31	81	57
D	71	43	41	47
E	52	29	51	50

Find by using the assignment method, the assignment of machinists to jobs that will result in a maximum profit. Which job should be declined.
[Ans. $A \to$ dummy, $B \to 2$, $C \to 3$, $D \to 1$, $E \to 4$; max profit = 256] **[JNTU (Mech) 99, 98]**

4. A company has six jobs to be processed by six mechanics. The following table gives the return in rupees when the ith job is assigned to the jth mechanic (i; $j = 1, \ldots, 6$). How should the jobs be assigned to the mechanics so as to maximize the overall return.

		Job					
		I	II	III	IV	V	VI
	1	9	22	58	11	19	27
	2	43	78	72	50	63	48
Mechanic	3	41	28	91	37	45	33
	4	74	42	27	49	39	32
	5	36	11	57	22	25	18
	6	3	56	53	31	17	28

[Meerut B.Sc. (Math.) 90 ; M.Sc. Baroda B.Sc. (Math.) 81]
[Ans. $1 \to VI$, $2 \to V$, $2 \to III$, $4 \to I$, $5 \to IV$, $6 \to II$, max. return = 333 .

5. Five lathes are to be allotted to five operators (one for each). The following table gives weekly output figures (in pieces) :

		Weekly Output in Lathe				
		L_1	L_2	L_3	L_4	L_5
	P	20	22	27	32	36
	Q	19	23	29	34	40
Operator	R	23	28	35	39	34
	S	21	24	31	37	42
	T	24	28	31	36	41

Profit per piece is Rs. 25. Find the maximum profit. **[C.A. (Nov.) 93]**
[Hint : The given problem is of maximization one. So convert it into an opportunity loss matrix by subtracting all the elements from the highest element 42.]
[Ans. : $p \to L_1$, $Q \to L_5$, $R \to L_3$, $S \to L_4$, $T \to L_2$. Max. weekly output = 160 pieces.
Maximum profit = 25 × 160 = Rs. 4000.]

12·7-2 Restrictions on Assignment

Sometimes technical, legal or other restrictions do not permit the assignment of a particular facility to a particular job. Such difficulty can be overcome by assigning a very high cost (say, infinite cost) to the corresponding cell, so that the activity will be automatically excluded from the optimal solution. The following example will make the procedure clear.

Q. 1. How will you solve an assignment problem, where a particular assignment is prohibited.
2. What do you understand by restricted assignments ? **[JNTU (B. Tech.) 2003]**

Example 18. *A job shop has purchased 5 new machines of different type. There are 5 available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centres which would have a heavy work flow to and from these machines. Therefore, the objective is to assign the new machines to the available locations in order to minimize the total cost of material handling. The estimated cost per unit time of materials handling involving each of the machines is given below for the respective locations. Locations 1, 2, 3, 4 and 5 are not considered suitable for machines A, B, C, D and E, respectively. Find the optimal solution :*

Location (Cost in Rs.)

		1	2	3	4	5
	A	×	10	25	25	10
	B	1	×	10	15	2
Machine	C	8	9	×	20	10
	D	14	10	24	×	15
	E	10	8	25	27	×

How would the optimal solution get modified if location 5 is also unsuitable for machine A ?

Solution. Since locations 1, 2, 3, 4 and 5 are not suitable for machines A, B, C, D and E respectively, an extremely large cost (say ∞) should be attached to these locations. Then the cost matrix of the resulting assignment problem becomes as shown below :

∞	10	25	25	10
1	8	10	15	2
8	9	∞	20	10
14	10	24	∞	15
10	8	25	27	∞

∞	2	6	3	$\boxed{0}$
$\cancel{0}$	∞	$\boxed{0}$	2	1
$\cancel{0}$	3	∞	$\boxed{0}$	2
2	$\boxed{0}$	3	∞	3
$\boxed{0}$	$\cancel{0}$	6	5	∞

Following the usual procedure of solving an assignment problem, the optimum assignment is obtained as shown in the above right side table. This gives the optimal assignment as :

$A \to 5$, $B \to 3$, $C \to 4$, $D \to 2$, and $E \to 1$, with total min cost = Rs. 60 .

Now if location 5 is also not suitable for the machine A, we attach an extremely large cost (= ∞) to cell (1, 5). Again applying the assignment procedure to this modified problem, the following assignment solution can be easily obtained.

$$\left.\begin{array}{l} A \to 4,\ B \to 3,\ C \to 5,\ D \to 2,\ \text{and } E \to 1 \\ \text{or} \quad A \to 2,\ B \to 3,\ C \to 4,\ D \to 5,\ \text{and } E \to 1 \end{array}\right\} \text{ with min cost of Rs. 65.}$$

Q. Explain how to modify an effectiveness matrix in an assignment problem if a particular assignment is prohibited.

EXAMINATION PROBLEMS

1. Five operators have to be assigned to five machines. The assignment costs are given in the table below :

		Machine				
		I	II	III	IV	V
	A	5	5	—	2	6
	B	7	4	2	3	4
Operator	C	9	3	5	—	3
	D	7	2	6	7	2
	E	6	5	7	9	1

Operator A cannot operate machine III and operator C cannot operate machine IV. Find the optimal assignment schedule.
[Ans. A → IV , B → III , C → II , D → I , E → V ; or A → IV , B → III , C → V , D → II , E → I , min cost = 15 .]

2. Four new machines M_1, M_2, M_3 and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited waiting space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The assignment cost c_{ij} of machine i to place j, in rupees, is shown below :

	A	B	C	D	E
M_1	9	11	15	10	11
M_2	12	9	—	10	9
M_3	—	11	14	11	7
M_4	14	8	12	7	8

Obtain the optimal solution using best starting solution. **[Meerut (M. Com.) Jan. 98 BP]**

3. Four operators O_1, O_2, O_3 and O_4 are available to a manager who has to get four jobs J_1, J_2, J_3 and J_4 done by assigning one job to each operator. Given the times needed by different operators for different jobs in the matrix below :

	J_1	J_2	J_3	J_4
O_1	12	10	10	8
O_2	14	12	15	11
O_3	6	10	16	4
O_4	8	10	9	7

(i) How should the manager assign the jobs so that the total time needed for all four jobs is minimum ?

(ii) If job J_2 is not to be assigned to operator O_2, what should be the assignment and how much additional total time will be required ? **[C.A. (May) 94]**

[Ans. (i) $O_1 \to J_3$, $O_2 \to J_2$, $O_3 \to J_4$, $O_4 \to J_1$, min. time = 34.
(ii) $O_1 \to J_2$, $O_2 \to J_4$, $O_3 \to J_1$, $O_4 \to J_3$, min time = 36. } $\therefore$ Additional time required = 36 − 34 = 2 units of time.

4. Five swimmers are eligible to complete in a relay team which is to consist of four swimmers swimming four different swimming styles; back stroke, breast stroke, free style and butterfly. The time taken for the five swimmers—Anand Bhasker, Chandru, Dorai and Easwar—to cover a distance of 100 meters in various swimming styles are given below in minutes, seconds.

Anand swims the back stroke in 1 : 09, the breast stroke in 1 : 15 and has never competed in the free style or butterfly.

Bhasker is a free style specialist averaging 1 : 01 for the 100 meters but can also swim the breast stroke in 1 : 16 and butterfly in 1 : 20.

Chandru swims all styles—back 1 : 10, butterfly 1 : 12, free style 1 : 05 and breast stroke 1 : 20.

Dorai swims only the butterfly 1 : 11 while Easwar swims the back stroke 1 : 20, the breast stroke 1 : 16, the free style 1 : 06 and the butterfly 1 : 10.

Which swimmer should be assigned to which swimming style ? Who will not be in the relay. **[C.A. (Nov.) 91]**

[Hint. The assignment matrix with time expressed in seconds and adding a dummy style to balance it is given by

	Back stroke	Breast stroke	Free style	Butterfly	Dummy
Anand	69	75	–	–	0
Bhasker	–	76	61	80	0
Chandru	70	80	65	72	0
Dorai	–	–	–	71	0
Easwar	80	76	66	70	0

[Ans. Anand will be in Breast stroke, (time 75 secs.)
Bhasker will be in free stroke, (time 61 secs.)
Chandru will be in Back stroke, (time 70 secs.)
Dorai will not participate (dummy), (time 70 secs.)
Easwar will be in Butterfly, (time 70 secs.)
Dorai will be out of the relay.
Total minimum time in the relay (= 276 secs or 4 min. 36 sec.)

5. WELLDONE company has taken the third floor of a multistoreyed building for rent with a view to locate one of their zonal offices. There are five main rooms in this floor to be assigned to five managers. Each room has its own advantages and disadvantages. Some have windows, some are closer to washrooms or to the canteen or secretarial pool. The rooms are of all different sizes and shapes. Each of the five managers were asked to rank their room preferences amongst the rooms 301, 302, 303, 304 and 305. Their preferences were recorded in a table as indicated below.

M_1	M_2	M_3	M_4	M_5
302	302	303	302	301
303	304	301	305	302
304	305	304	304	304
	301	305	303	
		302		

Most of the managers did not list all the five rooms since they were not satisfied with some of these rooms and they have left off these from the list. Assuming that their preferences can be quantified by numbers, find out as to which manager should be assigned to which room so that their total preference ranking is a minimum. **[C.A. (Nov.) 90]**

[**Hint.** Formulation of preference ranking assignment problem is :

		M_1	M_2	M_3	M_4	M_5
	301	–	4	2	–	1
	302	1	1	5	1	2
Room No.	303	2	–	1	4	–
	304	3	2	3	3	3
	305	–	3	4	2	–

[**Ans.** $M_1 \to 302$, $M_2 \to 304$, $M_3 \to 303$, $M_4 \to 305$, $M_5 \to 301$, and the total minimum ranking is $1 + 2 + 1 + 2 + 1 = 7$?

6. Imagine yourself to be the *Executive Director* of a 5-star Hotel which has four banquet halls that can be used for all functions including weddings. The halls were all about the same size and the facilities in each hall differed. During a heavy marriage seen, 4 parties approached you to reserve a hall for the marriage to be celebrated on the same day. These marriage parties were told that the first choice among these 4 halls would cost Rs. 10,000 for the day. They were also required to indicate the second, third and fourth preferences and the price that they would be willing to pay marriage party A & D indicated that they won't be interested in Halls 3 & 4. Other particulars are given in the following table :

Revenue/Hall Table

Marriage Party	Hall			
	1	2	3	4
A	10,000	9,000	×	×
B	8,000	10,000	8,000	5,000
C	7,000	10,000	6,000	8,000
D	10,000	8,000	×	×

Where × indicates that the party does not want that Hall. Decide on an allocation that will maximize the revenue to your Hotel. **[C.A. (May) 95]**

[**Hint.** To solve this problem of maximization, first convert it to a minimization problem by subtracting all the elements of the given matrix from its highest element which is equal to Rs. 10,000 here. The matrix thus obtained will be named as loss matrix. Now apply assignment algorithm to the loss matrix.]

[**Ans.** Marriage party A → Hall 2, B → 3, C → 4 and D → 1.
Maximum revenue = Rs. (9,000 + 8,000 + 8,000 + 10,000) = Rs. 35,000.]

7. The secretary of a School is taking bids on the city's four school bus routes. Four companies have made the bids as detailed in the following table :

Company	Bids (in Rs.)			
	Route 1	Route 2	Route 3	Route 4
1	4,000	5,000	–	–
2	–	4,000	–	4,000
3	3,000	–	2,000	–
4	–	–	4,000	5,000

Suppose each bidder can be assigned only one route. Use the assignment model to minimize the school's cost of running the four bus routes. **[C.A. (Nov.) 95]**

[**Hint.** Since some of the companies have not made bids for certain routes, assign a very high bid M for all such routes. Then apply assignment algorithm.]

[**Ans.** Company 1 → Route 1, Company 2 → Route 2, Company 3 → Route 3, Company 4 → Route 4. Minimum cost = Rs. (4000 + 4000 + 2000 + 5000) = Rs. 15,000.]

8. Suggest optimum assignment of 4 workers *A*, *B*, *C* and *D* to 4 jobs, I, II, III and IV. The time taken by different workers in completing the different jobs is given below :

		Job I	Job II	Job III	Job IV
	A	8	10	12	16
Worker	B	11	11	15	8
	C	9	6	5	14
	D	15	14	9	7

Also indicate the total time taken in completing the jobs. **[Raj. (M. Com.) 98; Delhi (M. Com.) 96]**

[**Ans.** Optimum assignment is :

$$I \xrightarrow{8} A,\ II \xrightarrow{11} B,\ III \xrightarrow{5} C,\ IV \xrightarrow{7} D, \text{ Min cost } = 31\ ]$$

9. A construction company has requested bids for subcontracts on fire different projects. Five companies have responded, their bids are represented below. Determine the minimum cost assignment of subcontracts to bidders, assuming that each bidder can receive only one contract.

Bid amount (000's Rs.)

		I	II	III	IV	V
	1	41	72	39	52	25
	2	22	29	49	65	81
Bidder	3	27	39	60	51	40
	4	45	50	48	52	37
	5	29	40	45	26	30

[Delhi (M.B.A.) Nov. 95]

10. The XYZ company has 5 jobs *I, II, III, IV V* to be done and 5 men *A, B, C, D, E* to do these jobs. The number of hours each man would take to accomplish each job is given by the following table.

	L	*M*	*N*	*O*	*P*
A	16	13	17	19	20
B	14	12	13	16	17
C	14	11	12	17	18
D	5	5	8	8	11
E	5	3	8	8	10

Work out the optimum assignment and the total minimum time taken. **[Allahabad (M.B.A.) Feb., 99]**

[**Ans.** $A \xrightarrow{13} M,\ B \xrightarrow{17} P,\ C \xrightarrow{12} N,\ D \xrightarrow{5} L,\ E \xrightarrow{8} O$ cost = 55]

11. Given the following data, determine the least cost allocation of the available machines to five jobs.

		Job				
		A	*B*	*C*	*D*	*E*
	·	25	29	31	42	37
	2	22	19	35	18	26
Machine	3	39	38	26	20	33
	4	34	27	28	40	32
	5	24	42	36	23	45

[Andhra (M.B.A.) 98]

[**Ans.** $1 \xrightarrow{25} A,\ 2 \xrightarrow{19} B,\ 3 \xrightarrow{26} C,\ 4 \xrightarrow{32} E,\ S \xrightarrow{25} D$ cost = 125]

12. A company is producing a single product and is selling it through five agencies situated in different cities. All of a sudden, there is a demand for the product in another five cities not having any agency of the company. The company is faced with a problem of deciding on how to assign the existing agencies to despatch the product to needy cities in such a way that the travelling distance is minimized. The distance between the surplus and deficit cities (in kms) is given in the following table :

Deficit Cities

		a	*b*	*c*	*d*	*e*
	A	160	130	175	190	200
Surplus Cities	B	135	120	130	160	175
	C	140	110	125	170	185
	D	50	50	80	80	110
	E	55	35	80	80	105

Determine the optimum assignment schedule. **[Delhi (M.B.A.) Nov. 98]**

13. A company receives tenders for four projects from five contractors. Only one project can be assigned to any contractor. The tender received (in thousands of rupees) are given below. Contractor *D* does not want to carry out project 2 and has not, therefore, submitted the tender for that.

		Contractor				
		A	*B*	*C*	*D*	*E*
Project	1	500	600	150	450	600
	2	400	550	200	—	550
	3	450	575	175	425	610
	4	475	575	185	440	590

(i) Using the Hungarian method, find the set of assignments with the smallest possible total cost.
(ii) What will be the minimum cost of getting all the projects completed ? **[Delhi (M. Com.) 98]**

14. The Secretary of a School is taking bids on the City's four school bus routes. Four companies have made the bids as detailed in the following table :

	BIDS			
	Route I	Route 2	Route 3	Route 4
Company 1	Rs. 4,000	Rs. 5,000	—	—
Company 2	—	Rs. 4,000	—	Rs. 4,000
Company 3	Rs. 3,000	—	Rs. 2,000	—
Company 4	—	—	Rs. 4,000	Rs. 5,000

Suppose each bidder can be assigned only one route. Use the assignment model to minimize the school's cost of running the four bus routes. **[C.A. (Nov.) 95]**

[Hint.

Steps 1 : Row reduction

	BIDS			
	Route I	Route 2	Route 3	Route 4
Company 1	$\boxed{0}$	1,000	—	—
Company 2	—	$\boxed{0}$	—	0
Company 3	1,000	—	$\boxed{0}$	—
Company 4	—	—	0	1,000

Steps 2 : Since each column has a zero, the column reduction will give the same matrix. Let us start assigning as usual.

	Route I	Route 2	Route 3	Route 4
Company 1	0	1,000	—	—
Company 2	—	0	—	0
Company 3	1,000	—	0	—
Company 4	—	—	0	1,000

The minimum uncovered element is 1,000.

Step 3. Performing the usual operations, Step 4 : The desired solution is

	Route I	Route 2	Route 3	Route 4
Company 1	$\boxed{0}$	1,000	—	—
Company 2	—	$\boxed{0}$	—	0
Company 3	1,000	—	$\boxed{0}$	—
Company 4	—	—	0	$\boxed{0}$

Company	Route	Cost (Rs.)
1	1	4,000
2	2	4,000
3	3	2,000
4	4	5,000

15. An organisation producing 4 different products, *viz.*, *A*, *B*, *C* and *D* having 4 operators, *viz.*, *P*, *Q*, *R* and *S* who are capable of producing any of the four products, works effectively 7 hours a day. The time (in minutes) required for each operator for producing each of the products are given in the cells of the following matrix along with profit (Rs. per unit) :

	Product			
Operator	A	B	C	D
P	6	10	14	12
Q	7	5	3	4
R	6	7	10	10
S	20	10	15	15
Profit (Rs./unit)	3	2	4	1

Find out the assignment of operators to product which will maximize the profit. **[C.A. (May) 96]**

[**Hint.** Step 1 : Given that the unit (factory) works effectively for 7 hours and the processing time (in minutes) for each of the four products by different operators, we obtain the production and profit matrices as follows :

PRODUCTION MATRIX PROFIT MATRIX

Operator	Product A	B	C	D
P	70	42	30	35
Q	60	84	140	15
R	70	60	42	42
S	20	42	28	28

Operator	Product A	B	C	D
P	210	84	120	35
Q	180	168	560	105
R	210	120	168	42
S	60	84	112	28

To use the same algorithm for minimization, subtract all the elements from the highest value and obtain the following matrix :

Operator	Product A	B	C	D
P	350	476	440	525
Q	380	392	0	455
R	350	440	392	518
S	500	476	448	532

Subtracting row minimum, we obtain

Operator	Product A	B	C	D
P	0	126	90	175
Q	380	392	0	455
R	0	90	42	168
S	52	28	0	84

Subtracting column minimum and after assignment, we get

Operator	Product A	B	C	D
P	$\boxed{0}$	98	90	91
Q	380	364	$\boxed{0}$	371
R	0	62	42	84
S	52	$\boxed{0}$	0	—

Since required number of assignments could not be made, proceed further.

Operator	Product A	B	C	D
P	$\boxed{0}$	36	90	29
Q	380	302	$\boxed{0}$	309
R	0	$\boxed{0}$	42	22
S	114	0	62	$\boxed{0}$

The optimum solution is :

Operator	Product	Profit (Rs.)
P	A	210
Q	C	560
R	B	120
S	D	28
Total profit		918

16. A company has four sales representatives who are to be assigned to four different sales territories. The monthly sales increase estimated for each sales representative for different sales territories (in lakh rupees), are shown in the following table :

		Sales Territories I	II	III	IV
Sales Representative	A	200	150	170	220
	B	160	120	150	140
	C	190	195	190	200
	D	180	175	160	190

(i) Suggest optimum assignment and the total maximum sales increase per month.

(ii) If for certain reasons, sales representative *B* cannot be assigned to sales territory III, will the optimum assignment schedule be different ? If so, find that schedule and the effect on total sales. **[Delhi (M. Com.) 97]**

17. (a) A firm produces four products. There are four operators who are capable of producing any of these four products. The processing time varies from operator to operator. The firm records 8 hours a day and allows 30 minutes for lunch. The processing time in minutes and the profit for each of the product are given below :

Operator	Product			
	A	B	C	D
1	15	9	10	6
2	10	6	9	6
3	25	15	15	9
4	15	9	10	10
Profit (Rs. per unit)	8	6	5	4

Find the optimum assignment of product to operators. **[C.A. Nov., 97]**

(b) Five lathes are to be allotted to five operators (one for each). The following table gives weekly output figures (in pieces) :

	Lathes				
Operator	L_1	L_2	L_3	L_4	L_5
P	20	22	27	32	36
Q	19	23	29	34	40
R	23	28	35	39	34
S	21	24	31	37	42
T	24	28	31	36	41

Profit per piece is Rs. 25. Find the maximum profit per week.

[**Hint.** The net working time is 450 minutes per day. The number of items that could be produced by the four operators is given below :

Operator	Product			
	A	B	C	D
1	30	50	45	75
2	45	75	50	75
3	18	30	30	50
4	30	50	45	45

Multiplying with the corresponding profit, we obtain the following matrix for finding maximizing profit.

Operator	Product			
	A	B	C	D
1	240	300	225	300
2	360	450	250	300
3	144	180	150	200
4	240	300	225	180

FINAL TABLE : ***Optimum solution***

Operator	Product			
	A	B	C	D
1	0	0	21	$\boxed{0}$
2	30	$\boxed{0}$	146	150
3	0	24	$\boxed{0}$	4
4	$\boxed{0}$	0	21	120

The assignment is

Operator		Product	Profit
1	→	D	300
2	→	B	450
3	→	C	150
4	→	A	240
		Rs.	1,140

18. (a) Find an assignment of operators to machines that gives maximum production with the data given below :

		Machines			
		A	B	C	D
Operators	1	10	5	7	8
	2	11	4	9	10
	3	8	4	9	7
	4	7	5	6	4
	5	8	9	7	5

[AIMA (PG Dep. in Management) Dec. 97]

(b) Six salesman are to be allocated to six sales regions. The earnings of each salesman at each region is given below. How can you find an allocation so that earnings will be maximum.

Salesmen	Region					
	1	2	3	4	5	6
A	15	35	0	25	10	45
B	40	05	45	20	15	20
C	25	60	10	65	25	10
D	25	20	35	10	25	60
E	30	70	40	5	40	50
F	10	25	30	40	50	15

(Fig. given are in 000) **[AIMA (P.G. Dip. in Management), June 98]**

19. The marketing director of a multi-unit company is faced with a problem of assigning 5 senior managers to six zones. From the past experience he knows that the efficiency percentage judged by sales, operating cost etc. depends on manager-zone combination. the efficiency of different managers is given below :

	Zone					
Manager	I	II	III	IV	V	VI
A	73	91	87	82	78	80
B	81	85	69	76	74	85
C	75	72	83	84	78	91
D	93	96	86	91	83	82
E	90	91	79	89	69	76

Find out which zone will be managed by a junior manager due to non-availability of senior manager.

[Poona (M.B.A.) 98]

[**Ans.** $A \to III$, $B \to II$, $C \to VI$, $D \to I$, $E \to IV$, $F \to V$ and zone V should be managed by a junior manager.]

20. To stimulate interest and provide an atmosphere for intellectual discussion, a finance faculty in a management school decides to hold special seminars to four contemporary topics—leasing, portfolio management, private mutual funds, swaps and options. Such seminars should be held once in a week in the afternoons. However, scheduling these seminars (one for each topic, and not more than one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A cereful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows :

	Leasing	Portfolio management	Private mutual funds	Swaps and options
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimum schedule of the seminars. Also find out the total number of students who will be missing at least one seminar. **[C.A., May 99]**

[**Hint.** Final Table : Optimum Solution

		I	II	III	IV	V
		Leasing	Portfolio Management	Private Mutual Funds	Swaps and options	Dummy
1.	Mon.	30	20	40	0	0
2.	Tues.	20	10	20	10	10
3.	Wed.	40	0	10	0	0
4.	Thurs.	10	10	0	10	0
5.	Fri.	0	10	0	20	10

[**Ans.** $1 \xrightarrow{20} IV$, $2 \xrightarrow{0} V$, $3 \xrightarrow{20} II$, $4 \xrightarrow{20} III$, $4 \xrightarrow{10} I$, cost = 70.]

21. A company is considering an expansion into five new sales territories. The company has recruited four new salesmen. Based on the salesmen's experiences and personality the sales managers has assigned a rating to each of the salesmen for each of the sales territories. The ratings are as follows :

		Territories				
		1	2	3	4	5
Salesmen	A	75	80	85	70	90
	B	91	71	82	75	85
	C	78	90	85	80	80
	D	65	75	88	85	90

Suggest optimum assignment of the salesmen. If for certain reasons, saleman D cannot be assigned to territory 3, will the optimum assignment be different ? If so what would be the new assignment schedule ? **[Delhi (M.B.A.) 99]**

22. A fast-food chain wants to build four stores. In the past, the chain has used six different construction companies and, having been satisfied with each, has invited each to bid on each job. The final bids (in '000 rupees) were as shown in the following table :

	Construction company					
	1	2	3	4	5	6
Store 1	85·3	88	87·5	82·4	89·1	86·7
Store 2	78·9	77·4	77·4	76·5	79·3	78·3
Store 3	82	81·3	82·4	80·6	83·5	81·7
Store 4	84·3	84·6	86·2	83·3	84·4	85·5

Since the fast-food chain wants to have each of the new stores ready as quickly as possible, it will award at most one job to a construction company. What assignment results in minimum total cost to the fast-food chain ?

[Delhi (M.B.A.) Dec., 94]

23. The ABC ice cream company has a distribution depth in Greater Kailash Part I for distributing ice cream in South Delhi. There are four vendors located in different parts of South Delhi (call them A, B, C and D) who have to be supplied ice cream every day. The following matrix displays the distances in (km.) between the depot and the four vendors :

		To				
		Depot	Vendor A	Vendor B	Vendor C	Vendor D
From	Depot.	—	3·5	3	4	2
	Vendor A	3·5	—	4	2·5	3
	Vendor B	3	4	—	4·5	3·5
	Vendor C	4	2·5	4·5	—	4
	Vendor D	2	3	5·5	4	—

What route should the company van follow so that the total distance travelled is minimized ? **[Delhi (M.B.A.) April 96]**

[Hint. Final Table : Optimum Solution

		To				
		Depot	Vendor A	Vendor B	Vendor C	Vendor D
From	Depot.	—	1·5	0	2	$\boxed{0}$
	Vendor A	1·5	—	0·5	$\boxed{0}$	0·5
	Vendor B	$\boxed{0}$	0·5	—	1	0
	Vendor C	2	$\boxed{0}$	1	—	1·5
	Vendor D	0	0·5	$\boxed{0}$	1·5	—

24. *XYZ* Airline operating 7 days a week has given the following time-table. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing flights that minimizes layover time away from home. For any given pairing the crew will be based at the city that results in the smaller layover :

Chennai—Mumbai			Mumbai—Chennai		
Flight Number	Depart.	Arrive.	Flight Number	Depart.	Arrive.
A_1	6 AM	8 AM	B_1	8 AM	10 AM
A_2	8 AM	10 AM	B_2	9 AM	11 AM
A_3	2 PM	4 PM	B_3	2 PM	4 PM
A_4	8 PM	10 PM	B_4	7 PM	9 PM

[Hint. See Example 5.]

[Ans. $A_1 \to B_3$, $A_2 \to B_4$, $A_3 \to B_1$, $A_4 \to B_2$ *, Min. Layover time 40 hrs.] **[CA (May) 2000]**

25. Due to absence of a workman, an officer has to assign four out of five different jobs to four workers with the performance matrix given below :

		Operators			
		A	B	C	D
Jobs	1	3	6	5	3
	2	4	9	3	2
	3	11	2	4	6
	4	10	4	6	5
	5	11	12	14	10

[Ans. 1 → A, 2 → C, 3 → B, 4 → D, 5 – D' or 1 → A, 2 → D, 3 → C, 4 → B, 5 – D'] **[IGNOU 2000]**

26. Four different jobs can be done on four different machines. The set up and down time costs are prohibitively high for change overs. The matrix below gives the cost in hundreds of Rs. for job J_i to M_j.

		Machine			
		M_1	M_2	M_3	M_4
	J_1	10	13	9	15
Job	J_2	12	10	12	9
	J_3	16	14	15	13
	J_4	11	11	12	8

Assign the jobs to the machines in order to minimize the total cost. **[AIMS (Bang.) MBA 2002]**

27. Four trucks available in location are to be sent to 1, 2, 3 and 4 vacant spaces A, B, C, D, E and F so that the total distance travelled is minimized. The elements in the matrix below shows the distance in Km. Determine the optimal assignment of the trucks to the spaces. **[JNTU (B. Tech.) 2003]**

	1	2	3	4
A	4	7	3	7
B	8	2	5	5
C	4	9	6	9
D	7	5	4	8
E	6	3	5	4
F	6	8	7	3

28. A firm is contemplating the introduction of three products 1, 2 and 3, in its three plants A, B and C. Only a single product is decided to be introduced in each of the plants. The unit cost of producing a product in a plant, is given in the following matrix.

		Plant		
		A	B	C
	1	8	12	—
Product	2	10	6	4
	3	7	6	6

(a) How should the product be assigned so that the total unit cost is minimized ?

(b) If the quantity of different products to be produced is as follows, then what assignment shall minimize the aggregate production cost ?

Product	Quantity (in units)
1	2,000
2	2,000
3	10,000

(c) What would your answer be if the three products were to be produced in equal quantities ?

(d) It is expected that the selling prices of the products produced by different plants would be different as shown in the following table :

		Plant		
		A	B	C
	1	15	18	—
Product	2	18	16	10
	3	12	10	8

Assuming the quantities mentioned in (b) above would be produced and sold, how should the products be assigned to the plants to obtain maximum profit ? **(Delhi (MBA) 2000)**

[**Ans.** (a) $1 \to A$, $2 \to C$, $3 \to B$.

(b) Multiply the given matrix by the quantity in units, then for the cost matrix, obtain optimal assignment $1 - A$, $2 \to B$, $3 \to C$ and optimal total cost = Rs. 78,000.

(c) Same as (a).

(d) Obtain profit matrix by using the relationship : Profit = Total Sales – Total cost,
$1 \to B$, $2 \to C$, $3 \to D$ and maximum total profit = Rs. 74,000.]

29. Four engineers are available to design four projects. Engineer 2 is not competent to design the project B. Given the following time estimates needed by each engineer to design a given project, find how should the engineers be assigned to projects so as to minimize the total design time of four projects. **[JNTU (Mach and Prod.) May 2004]**

Projects

		A	B	C	D
Engineers	1	12	10	10	8
	2	14	Not Suitable	15	11
	3	6	10	16	4
	4	8	10	9	7

12.8. SENSITIVITY IN ASSIGNMENT PROBLEMS

The structure of assignment problem is of such a type that there is very little scope for sensitivity analysis. Modest altèrations in the conditions (such as one being able to do two jobs) can be considered by repeating the man's row and adding a dummy column to square up the matrix.

Addition of a constant throughout any row or column also makes no difference to this position of optimal assignment. However, sometimes equiproportionate change throughout a row or column can make a difference. So in reference to assignment problems there is no scope for altering the level of an assignment.

12.9. THE TRAVELLING-SALESMAN (ROUTING) PROBLEM

The travelling salesman problem is one of the problems considered as puzzles by the mathematicians.

Suppose a salesman wants to visit a certain number of cities alloted to him. He knows the distance (or cost or time) of journey between every pair of cities, usually denoted by c_{ij}, *i.e.*, from city i to city j. His problem is to select such a route that starts from his home city, passes through each city once and only once, and returns to his home city in the shortest possible distance (or at the least cost or in least time).

The problem may be classified in two forms :

1. **Symmetrical.** The problem is said to be *symmetrical* if the distance (or cost or time) between every pair of cities is independent of the direction of his journey.
2. **Asymmetrical.** The problem is said to be *asymmetrical,* if for one or more pair of cities, the distance (or cost or time) changes with the direction. For example, flying from *East* to *West* usually takes longer time than from *West* to *East* on account of prevailing winds. Similar is the case while going up-hill from city A to B instead of coming down-hill from city B to A.

   Furthermore, if number of cities is only two, obviously there is no choice. If number of cities become three, say A, B and C, one of them (say A) is the home base, then there are two possible routes :
   $A \to B \to C$ and $A \to C \to B$.

   For four cities A, B, C and D, there are 3 ! = 6 possible routes, *i.e.*
   $$A \to B \to C \to D, \quad A \to B \to D \to C, \quad A \to C \to B \to D,$$
   $$A \to C \to D \to B, \quad A \to D \to B \to C, \quad \text{and} \quad A \to D \to C \to B.$$

But, if number of cities is increased to 21, there are 20 ! = 2,402,902,008, 176,640,000 different routes. Even a fast electronic computer testing one route per micro second and working 8 hr a day 365 days a year, would take almost a quarter of a million years to find the best solution. Thus, it is practically impossible to find the best route by trying each one. In general, if there are n cities, there are $(n - 1)$! possible routes. At present, the best procedure is to solve the problem as if it were an assignment problem. It becomes necessary to formulate this type of sequencing problem in the form of an assignment problem with the additional restriction on his choice of route.

12·9-1. Formulation of Travelling-Salesman Problem as an Assignment Problem

Suppose c_{ij} is the *distance* (or *cost* or *time*) from city i to city j and $x_{ij} = 1$ if the salesman goes directly from city i to city j ; and $x_{ij} = 0$ otherwise. Then minimize $\sum_i \sum_j x_{ij} c_{ij}$ with the additional restriction that the x_{ij} must be so chosen that no city is visited twice before the tour of all cities is completed. In particular, he cannot go directly from city i to i itself. This possibility may be avoided in the minimization process by adopting the convention $c_{ij} = \infty$ which ensures that x_{ij} can never be unity.

Alternatively, omit the variable x_{ij} from the problem specification. It is also important to note that only single $x_{ij} = 1$ for each value of i and j . The *distance* (or *cost* or *time*) matrix for this problem is given in *Table* 12·71

Table 12·71

From \ To	A_1	A_2	...	A_n
A_1	∞	c_{12}	...	c_{1n}
A_2	c_{21}	∞	...	c_{2n}
:	:	:	:	:
A_n	c_{n1}	c_{n2}	...	∞

Q. 1. Write a short note on travelling salesman problem.
2. State the travelling salesman problem and formulate it as an assignment problem. **[Kerala B.Sc. (Math.) 90]**
3. Discuss in detail the 'Travelling Salesman Problem. **[Madurai B.Sc (Appl. Math.) 94]**
4. Explain clearly the travelling salesman problem and discuss the method of solving it.
5. What is a travelling salesman problem ? **[JNTU (B. Tech.) 2003]**

12·9-2. Solution Procedure

For solving such problems, detailed procedure is explained with reference to an equivalent example involving the order in which five products A_1 , A_2 , A_3 , A_4 and A_5 are processed over a production facility. To find an order in which products are produced so that the set-up costs are minimized, is an asymmetrical travelling-salesman problem. The change-over/set-up costs between products must be produced once and only once and production must return to the first product. The following examples will make the procedure clear.

Example 19. *Given the matrix of set-up costs, show how to sequence the production so as to minimize the set-up cost per cycle.* **[Meerut (M.Sc.) 93 P]**

Table 12·72

From \ To	A_1	A_2	A_3	A_4	A_5
A_1	∝	2	5	7	1
A_2	6	∝	3	8	2
A_3	8	7	∝	4	7
A_4	12	4	6	∝	5
A_5	1	3	2	8	∝

Solution. Consider the problem as an assignment problem.

Step 1. First, subtract the smallest element from each row and each column to get the reduced matrix in *Table* 12·73 .

Table 12·73

From \ To	A_1	A_2	A_3	A_4	A_5
A_1	∝	2	5	7	1
A_2	6	∝	3	8	2
A_3	8	7	∝	4	7
A_4	12	4	6	∝	5
A_5	1	3	2	8	∝

Although, zeros of this matrix (marked □) give a solution to the assignment problem, but this is not a solution of the travelling-salesman problem. This solution ($A_1 \rightarrow A_5$, $A_5 \rightarrow A_1$, $A_2 \rightarrow A_3$, $A_3 \rightarrow A_4$ and $A_4 \rightarrow A_2$) indicates to produce the products A_1, then A_5 and then again A_1, without producing the products A_2, A_3 and A_4, thereby violating the additional restriction of producing each product once and only once before returning to the first product.

Step 2. Again, examine the matrix for some of the 'next best' solutions to the assignment problem, and try to find out one solution which satisfies the additional restriction. The smallest element other than zero is 1, so try the effect of putting such an element in the solution.

Table 12·74

	A_1	A_2	A_3	A_4	A_5
A_1	∞	$\boxed{1}$	3	6	$\cancel{0}$
A_2	4	∞	$\boxed{0}$	6	$\cancel{0}$
A_3	4	3	∞	$\boxed{0}$	3
A_4	8	$\cancel{0}$	1	∞	$\boxed{1}$
A_5	$\boxed{0}$	2	$\cancel{0}$	7	∞

Start by making unity-assignment in the cell (1, 2) instead of zero-assignment in the cell (1, 5) as shown in the above matrix *Table* 12·74. Delete row 1 and column 2 and get the remaining (4 × 4) matrix.

Although, there is no solution to the assignment problem among zeros, it is easy to see the best solution lies in the marked '□' elements. Thus, the required solution of the problem is : $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow A_5 \rightarrow A_1$.

The 'cost' in the reduced matrix (Table 12·74) is 2 for this solution.

Table 12·75

	A_1	A_2	A_3	A_4	A_5
A_1	∞	$\boxed{1}$	3	6	$\cancel{0}$
A_2	4	∞	$\cancel{0}$	6	$\boxed{0}$
A_3	4	3	∞	$\boxed{0}$	3
A_4	8	$\cancel{0}$	$\boxed{1}$	∞	X
A_5	$\boxed{0}$	2	$\cancel{0}$	7	∞

Step 3. It is seen that any solution where the 'cost' exceeds 2 is not optimal. Now, only examine solutions containing the element 1 in the cell (4, 3) not so far used to see if a better solution exists. After deleting row 4 and column 3 from Table 12·75, the remaining (4 × 4) matrix does not have a solution among the zeros, and it is concluded that the minimum cost for the reduced matrix is 2. Hence, the most suitable sequence is $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow A_5$ and the minimum set-up cost $= 2 + 3 + 4 + 5 + 1 =$ Rs. 15.

EXAMINATION PROBLEM

Q. 1. A salesman wants to visit cities A, B, C, D and E. He does not want to visit any city twice before completing his tour of all the cities and wishes to return to the point of starting journey. Cost of going from one city to another (in Rs.) is shown in the table below. Find the least cost route.

Note. For table see Example 19 and ∞ may be replaced by 0. **[VTU (BE Mech.) 2002]**

2. What is the difference between an assignment problem and a travelling salesman problem ? **[JNTU (B. Tech.) 2003]**

12·9-3 More Solved Examples

Example 20. *A salesman estimates that the following would be the cost on his route, visiting the six cities as shown in the table below :*

		To city					
		1	2	3	4	5	6
	1	∞	20	23	27	29	34
	2	21	∞	19	26	31	24
From City	3	26	28	∞	15	36	26
	4	25	16	25	∞	23	18
	5	23	40	23	31	∞	10
	6	27	18	12	35	16	∞

The salesman can visit each of the cities once and only once. Determine the optimum sequence he should follow to minimize the total distance travelled. What is the total distance travelled ? **[Meerut M.Sc. (Math.) 92]**

Solution. Step 1. Apply usual procedure of assignment algorithm to obtain the table showing an optimum assignment solution indicated by *marked zeros in* the following table :

	1	2	3	4	5	6
1	∞	$\cancel{0}$	$\boxed{0}$	7	2	14
2	$\boxed{0}$	∞	$\cancel{0}$	10	8	8
3	6	13	∞	$\boxed{0}$	14	11
4	4	$\boxed{0}$	6	∞	$\cancel{0}$	2
5	8	30	10	21	∞	$\boxed{0}$
6	13	6	$\cancel{0}$	26	$\boxed{0}$	∞

This table gives the optimum assignment solution $1 \to 3 \to 4 \to 2 \to 1$ and $5 \to 6 \to 5$ with min. distance 101. But, this table does not provide the solution to the travelling salesman problem, as it is not allowed to go from city 2 to 1 without visiting the cities 5 and 6.

Step 2. Once and only once, we try to find the 'next best' solution which satisfies the additional restriction. The smallest element other than zero is 2. So, we try to bring 2 into the solution. Since the element 2 occurs at two places, we shall consider both the cases saparately until the acceptable solution is attained.

We start making assignment with the cell (1, 5) having the next minimum element 2 instead of 0 in cell (1, 3). After making this assignment, we observe that no other assignment can be made in the first row and fifth column, and thus the resulting feasible solution will be $1 \to 5 \to 6 \to 3 \to 4 \to 2 \to 1$. The assigned elements for this solution are marked in the following table. The cost corresponding to this feasible solution is 2.

	1	2	3	4	5	6
1	∞	$\cancel{0}$	$\cancel{0}$	7	2	14
2	$\boxed{0}$	∞	$\cancel{0}$	10	8	8
3	6	13	∞	$\boxed{0}$	14	11
4	4	$\boxed{0}$	6	∞	$\cancel{0}$	2
5	8	30	10	21	∞	$\boxed{0}$
6	13	6	0	26	$\cancel{0}$	∞

Step 3. Again, if we make assignment in the cell (4, 6) having the next best element 2 instead of 0 marked in cell (4, 2), then no feasible solution is obtained in terms of zeros. Hence the best solution is :

$$1 \to 5 \to 6 \to 3 \to 4 \to 2 \to 1$$

The total set-up cost according to this route is 103.

Example 21. *Solve the travelling-salesman problem given by the following data*

$$C_{12} = 20\ .\ C_{13} = 4\ ,\ C_{14} = 10\ ,\ C_{23} = 5,\ C_{34} = 6,$$
$$C_{25} = 10\ ,\ C_{35} = 6\ ,\ C_{45} = 20\ ,\ \textit{where}\ C_{ij} = C_{ji},$$

and there is no route between cities i and j if the value for C_{ij} is not shown.

[Meerut 2005; Garhwal 97; Agra 93; IAS (Main) 91]

Solution. ***Step 1.*** First express the given problem in the form of an assignment problem by taking $C_{ij} = \infty$, for $i = j$ (Table 12·76).

Step 2. Subtract smallest element in each row from each element of that row. Then subtract smallest element in each column from each element of that column (Table 12·77).

Table 12·76

∞	20	4	10	∞
20	∞	5	∞	10
4	5	∞	6	6
10	∞	6	∞	20
∞	10	6	20	∞

Table 12·77

∞	15	$\boxed{0}$	4	∞	
15	∞	$\cancel{0}$	∞	3	
$\boxed{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$	L_2
4	∞	$\cancel{0}$	∞	12	
∞	3	$\cancel{0}$	12	∞	
		L_1			

Step 3. Now make zero-assignments (as shown by □) and cross-out other zeros in that row or column.

Step 4. (i) Draw minimum number of lines (L_1 and L_2) to cover all zeros (*Table* 12·77).

(ii) Pick up minimum element among all uncovered elements (which is 3 here).

(iii) Subtract 3 from all uncovered elements and add at the intersection of two lines.

Thus *Table* 12·78 is obtained.

Table 12·78

∞	12	$\boxed{0}$	1	∞
12	∞	$\cancel{0}$	∞	$\boxed{0}$
$\boxed{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$
1	∞	$\cancel{0}$	∞	9
∞	$\boxed{0}$	$\cancel{0}$	9	∞

L_4 L_3 L_2 L_1

Step 5. Again make zero-assignments in *Table* 12·78 and repeat the *step 4.*

Step 6. Minimum number of lines to cover all zeros is four here, and minimum element among all uncovered elements is 1. So subtract 1 from all uncovered elements and add 1 at the intersection of the lines. Thus *Table* 12·79 is obtained.

Table 12·79

	1	2	3	4	5
1	∞	12	$\boxed{0^*}$	$\cancel{0}$	∞
2	11	∞	$\cancel{0}$	∞	$\boxed{0^*}$
3	$\cancel{0}^*$	$\boxed{1}$	∞	$\boxed{0^*}$	1
4	$\boxed{0^*}$	∞	$\cancel{0}$	∞	[illegible]
5	∞	$\cancel{0}$	$\cancel{0}$	8	∞

Step 7. Make zero-assignment in *Table* 12·79 . Here, the solution of the assignment problem is :

$$(1, 3), (3, 4), (4, 1), (5, 2), (2, 5).$$

This is not the solution of the travelling-salesman problem as the sequence obtained is not in the cyclic order.

Step 8. It is seen that the next lowest number (other than zero) is 1. Therefore, instead of making zero assignment in the cell (5, 2), make assignment in the cell (3, 2) having the element 1.

Consequently, make assignment in the cell (5, 4) having element 8, instead of zero assignment in the cell (3, 2). Thus following assignment table is obtained having shortest route for the travelling-salesman, *i.e.* $1 \to 3 \to 2 \to 5 \to 4 \to 1$. **Ans.**

Table 12·80

	1	2	3	4	5
1	∞	12	$\boxed{0^*}$	$\cancel{0}$	∞
2	11	∞	$\cancel{0}$	∞	$\boxed{0^*}$
3	$\cancel{0}$	$\boxed{1^*}$	∞	$\cancel{0}$	1
4	$\boxed{0^*}$	∞	$\cancel{0}$	∞	9
5	∞	$\cancel{0}$	$\cancel{0}$	$\boxed{8^*}$	∞

EXAMINATION PROBLEMS
(On Travelling Salesman Problem)

1. Solve the following 'Travelling Salesman Problem' given by the following data :
$C_{12} = 4$, $C_{13} = 7$, $C_{14} = 3$, $C_{23} = 6$, $C_{24} = 3$ and $C_{34} = 7$ where $c_{ij} = c_{ji}$. **[IAS (Main) 91]**

2. A medical representative has to visit five stations A, B, C, D and E. He does not want to visit any station twice before completing his tour of all the stations, and wishes to return to the starting station. Costs of going from one station to another are given below. Determine the optimal route :

	A	B	C	D	E
A	∞	2	4	7	1
B	5	∞	2	8	2
C	7	6	∞	4	6
D	10	3	5	∞	4
E	1	2	2	8	∞

[Ans. $A \to B \to C \to D \to E \to A$]

3. Solve the travelling-saleman problem in the matrix shown below :

		To				
		1	2	3	4	5
	1	∞	6	12	6	4
	2	6	∞	10	5	4
From	*3*	8	7	∞	11	3
	4	5	4	11	∞	5
	5	5	2	7	8	∞

[**Ans.** 3 → 5 → 2 → 4 → 1 → 1 → 3, $z = 27$]

4. A salesman has to visit five cities *A, B, C, D,* and *E*. The distances (in hundred miles) between the five cities are as follows :

		To				
		A	B	C	D	E
	A	–	7	6	8	4
	B	7	–	8	5	6
From	*C*	6	8	–	9	7
	D	8	5	9	–	8
	E	4	6	7	8	–

If the salesman starts from city *A* and has to come back to city *A* , which route should he select so that the total distance travelled is minimum. **[JNTU (B. Tech.) 2003; Meerut (O. R.) Type 90]**

[**Ans.** $A \to E \to B \to D \to C \to A$, min. distance = 30 hundred miles]

5. A machine operator processes four types of item on his machine and must choose a sequence for them. The set-up cost per change depends on the items presently on machine and the set-up be made according to the following table :

		To item			
		A	B	C	D
	A	∞	4	7	3
From item	B	4	∞	6	3
	C	7	6	∞	7
	D	3	3	7	∞

If he processes each type of item once and only once each week, how should he sequence the items on this machine ? Use the method for the problem of travelling-salesman.

[**Ans.** (*i*) A → D → B → C → A, (*ii*) A → C → B → D → A; total min. cost = 19.]

6. A machine operator processes five types of items on his machine each week, and must choose sequence for them. The set-up cost per change depends on the items presently on the machine and item to be made, according to the following table :

		To item				
		A	B	C	D	E
	A	∞	4	7	3	4
	B	4	∞	6	3	4
From item	C	7	6	∞	7	5
	D	3	3	7	∞	7
	E	4	4	5	7	∞

If he produces each type of item once and only once each week, how should be sequence the items on his machine in order to minimize the total set up cost. **[Meerut (Maths) Jan. 98 BP]**

[**Ans.** A → E → C → B → D → A; min. cost = Rs. 21.1]

7. Products 1, 2, 3, 4 and 5 are to be processed on a machine. The set up costs in rupees per change depend upon the product presently on the machine and the set up to be made and are given as follows : $C_{12} = 16$, $C_{13} = 4$, $C_{14} = 12$, $C_{23} = 6$, $C_{34} = 5$, $C_{25} = 8$, $C_{35} = 6$, $C_{45} = 20$, $C_{ij} = C_{ji}$ and $C_{ij} = \infty$ for all values of *i* and *j* not given in the data. Find the optimum assignment of products in order to minimize the total set up costs. **[AIMS (ind.) Banglore 2002]**

8. Solve the following traveling salesman problem.

		To			
		A	B	C	D
	A	—	46	16	40
From	B	41	—	50	40
	C	82	32	—	60
	D	40	40	36	—

[JNTU (B. Tech.) 2003]

EXAMINATIONS REVIEW PROBLEMS

1. (a) What is Assignment Problem ? Give two areas of its applications.
   (b) How far sensitivity analysis is relevent to assignment.
   (c) Alpha corporation has four plants each of which can manufacture any one of four products. Production costs differ from one plant to another as do sales revenue. Given the revenue and cost data below, obtain which product each plant should produce to maximize profit.

Sales revenue (Rs.' 000s) Product

Plant	1	2	3	4
A	50	68	49	62
B	60	70	51	74
C	55	67	53	70
D	58	65	54	69

Production costs (Rs.' 000s)

Plant	1	2	3	4
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

[**Hint.** Construct the profit matrix by using the fact : Profit matrix = Revenue matrix – Cost matrix.
To make use of minimization technique, subtract each element of profit matrix from the maximum element which will be 8. Then apply assignment rule in usual manner.]
[Ans. *A* – *2*, *B* – *4*, *C* – *1*, *D* – *3*.]

2. Find the optimal solution for the assignment problem with the following cost matrix.

(i)

	I	II	III	IV	V
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

[JNTU (Mech. & Prod.) 2004; I.A.S. (Main) 2000]

(ii)

	I	II	III	IV
A	5	3	1	8
B	7	9	2	6
C	6	4	5	7
D	5	7	7	6

[JNTU (B. Tech.) 2003]

[Ans. (i) *A* – *I*, *B* – *IV*, *C* – *V*, *D* – *III*, *E* – *II*, min cost = 60, (ii) *A–III, B–IV, C–II, D–I*, min cost = 16]

5. A national truck-rental service has a surplus of one truck in each of the cities 1, 2, 3, 4, 5 and 6 : and a deficit of one truck in each of the cities 7, 8, 9, 10, 11 and 12. The distances (in kilometers) between the cities with a surplus and cities with a deficit are displayed below :

From \ To	7	8	9	10	11	12
1	31	62	29	42	15	41
2	12	19	39	55	71	40
3	17	29	50	41	22	22
4	35	40	38	42	27	33
5	19	80	29	16	2	23
6	72	30	30	50	41	20

How should the trucks be dispersed so as to minimize the total distance travelled ?

6. Solve the following assignment problems :

(i)

Tasks \ Men	I	II	III	IV	V
A	1	3	2	8	8
B	2	4	3	1	5
C	5	6	3	4	6
D	3	1	4	2	2
E	1	5	6	5	4

[Meerut 2005]

[Ans. *A* – *I*, *B* – *IV*, *C* – *III*, *D* – *II*, *E* – *V*].

(ii)

Tasks \ Persons	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

[Ans. $A \rightarrow 2$, $B \rightarrow 3$, $C \rightarrow 4$, $D \rightarrow 1$ min cost = 38].

(iii)

	1	2	3	4
I	12	30	21	15
II	18	33	9	31
III	44	25	24	21
IV	23	30	28	14

[Ans. *I* – 1, *II* – 3, *III* – 2, *IV* – 4]

(iv)

	I	II	III	IV
1	2	3	4	5
2	4	5	6	7
3	7	8	9	8
4	3	5	8	4

[Meerut (Maths) 91]

[Ans. (i) 1 – *I*, 2 – *II*, 3 – *III*,
(ii) 1 – *II*, 2 – *I*, 3 – *IV*, 4 – *III* : and others]

(v) Man

Jobs	1	2	3	4	5
I	12	8	7	15	4
II	7	9	17	14	10
III	9	6	12	6	7
IV	7	6	14	6	10
V	9	6	12	10	6

[Ans. (i) I – 3 , II – 1 , III – 2 , IV – 4 , V – 5 ;
(ii) I – 3 , II – 1 , III – 4 , IV – 2 , V – 5].

(vi) Job

Person	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

[IAS (Maths) 89]
[Ans. $A \to 5$, $B \to 1$, $C \to 4$, $D \to 3$, $E \to 2$, min. cost = 9]

(vii)

	I	II	III	IV	V
A	15	21	17	4	9
B	3	40	21	10	7
C	9	6	5	8	10
D	14	8	6	9	3
E	21	16	18	7	4

[Ans. A – V , B – I , C – II , D – III , E – V].

(viii)

	I	II	III	IV	V
A	6	5	8	11	16
B	1	13	16	1	10
C	16	11	8	8	8
D	9	14	12	10	16
E	10	13	11	8	16

[Ans. (i) $A \to II$, $B \to I$, $C \to V$, $D \to III$, $E \to IV$;
(ii) $A \to II$, $B \to IV$, $C \to V$, $D \to 1$, $E \to III$, min cost = 34].

7. There are five jobs to be assigned one each to 5 machines and the associated cost matrix is as follows :

Machine

Job	1	2	3	4	5
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

[Ans. A – 1 , B – 4 , C – 5 , D – 3, E – 2 min cost = Rs. 60].

8. An air freight company picks up and delivers freight where customers require. Company has two types of aircrafts *X* and *Y* with equal loading capacities but different operating costs. These are shown below :

Type of aircraft	Cost per freight	Freight (Rs.)
	Empty	Loaded
X	1·00	2·00
Y	1·50	3·00

The present four locations of the aircrafts which the company is having are as shown below :

$$J \to X, \ K \to Y, \ L \to Y; \ M \to X$$

Four customers of the company located at *A, B, C* and *D* want to transport nearly the same size of load to their final destinations. The final destinations are at a distance of 600, 300, 1000 and 500 kms from the loading points *A, B, C* and *D*, respectively.

Distances in kms. between location of the aircraft and the loading points are as follows .

Loading points

Present location of aircraft	A	B	C	D
J	300	200	400	100
K	300	100	300	300
L	400	100	100	500
M	200	200	400	200

Determine the allocations which minimize the total cost of transportation.
[Ans. $J \to D$; $K \to B$, $L \to C$, $M \to A$].

9. The jobs *A, B, C* are to be assigned three machines *X, Y, Z.* The processing costs (Rs.) are as given in the matrix shown below. Find the allocation which will minimize the overall processing cost.

Machine

Job	X	Y	Z
A	19	28	31
B	11	17	16
C	12	15	13

[Ans. $A-X$; $B-Y$, $C-Z$].

10. A project work consists of four major jobs for which four contractors have submitted tenders. The tender amounts quoted in lakhs of rupees are given in the matrix below :

Job

Contractor	a	b	c	d
1	10	24	30	15
2	16	22	28	12
3	12	20	32	10
4	9	26	34	16

Find the assignment which minimizes the total cost of project [each contract has to be assigned at least one job].
[Ans. Three alternative assignments are : (i) $1 \to b$, $2 \to c$, $3 \to d$, $4 \to a$ (ii) $1 \to c$, $2 \to b$, $3 \to d$, $4 \to a$; (iii) $1 \to c$, $2 \to d$, $3 \to b$, $4 \to a$; min. cost = Rs. 71,00,000].

11. A company is producing a single product and selling it through five agencies situated in different cities. All of a sudden, there is a demand for the product in another five cities not having any agency of the comapny. The company is faced with the problem of deciding on how to assign the existing agencies to despatch the product to needy cities in such a way that the total travelling distance is minimized. The distance between the surplus and deficit cities (in kilometers) is given by

Deficit city

Surplus city	A′	B′	C′	D′	E′
A	10	5	9	18	11
B	13	19	6	12	14
C	3	2	4	4	5
D	18	9	12	17	15
E	11	6	14	19	10

Determine the optimum assignment schedule.

[Ans. $A \to A'$, $B \to C'$, $C \to D'$, $D \to B'$, $E \to E'$, $z^* = 39$ km.]

12. Find the minimum cost solution for the 5 × 5 assignment problem whose cost coefficients are as given below :

	1	2	3	4	5
1	−2	−4	−8	−6	−1
2	0	−9	−5	−5	−4
3	−3	−8	0	−2	−6
4	−4	−3	−1	0	−3
5	−9	−5	−9	−9	−5

[Ans. $1 \to 3$, $2 \to 2$, $3 \to 5$, $4 \to 4$, $5 \to 1$ or $1 \to 4$, $2 \to 2$, $3 \to 3$, $4 \to 5$, $5 \to 1$, $z^* = 36$]

13. A company has 4 machines of which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table :

Machine

Job	W	X	Y	Z
A	18	24	28	32
B	8	13	17	18
C	10	15	19	22

What are the job-assignments which will minimize the cost ?

[Ans. $A \to W$, $B \to X$, $C \to Y$, or $A \to W$, $B \to Y$, $C \to X$, $z^* = 50$].

14. Six wagons are available at six stations *A, B, C, D, E* and *F.* These are required at stations *I, II, III, IV, V* and *VI.* The mileage between various stations is given by the following table :

	I	II	III	IV	V	VI
A	20	23	18	10	16	20
B	50	20	17	16	15	11
C	60	30	40	55	8	7
D	6	7	10	20	100	9
E	18	19	28	17	60	70
F	9	10	20	30	40	55

How should the wagons be transported in order to minimize the total mileage covered.
[Ans. $A \to IV$, $B \to VI$, $C \to V$, $D \to III$, $E \to I$, $F \to II$, total mileage = 66].

15. The owner of a small machine shop has four machinists avilable to assign to jobs for the day. five jobs offered with the expected profit (in Rs.) for each machinist on each job being as follows.

		Job				
		A	B	C	D	E
Machinist	1	6.20	7.80	5.00	10.10	8.20
	2	7.10	8.40	6.10	7.30	5.90
	3	8.70	9.20	11.10	7.10	8.10
	4	4.80	6.40	8.70	7.70	8.00

Find the assignment of machinists to jobs that will result in a maximum profit. Which job should be declined ?
[Ans. $1 \to D$, $2 \to B$, $3 \to C$, $4 \to E$, $5 \to A$, min. cost = Rs. 37·60. Job A should be declined].

16. A computer centre has got three programmers. The centre needs three application programmes to be developed. The Head of the computer centre, after studying carefully the programmes to be developed, estimates the computer time in minutes required by the experts tc the application programmes as follows :

		Programmers		
		A	B	C
Programmes	1	120	100	80
	2	70	90	110
	3	110	140	120

Assign the programmers to the programmes in such a way that the total computer time is least.
[Ans. $1 \to C$, $2 \to B$, $3 \to A$]

17. The Research & Development department of an organization is having four major jobs to be completed in ensuing financial period. There are four subgroups who can work on these jobs. Because of the technical nature of problems and heterogeneous combination of groups the cost of completing the work is different for different groups as shown in the following table : **[JNTU (Mech. & Prod.) 2004]**

		Job			
		1	2	3	4
Group	I	20	22	28	15
	II	16	20	12	13
	III	19	23	14	25
	IV	10	16	12	10

Allocate the jobs to the groups in such a way that the R & D budget is minimum.
[Ans. $i \to 2$, $II \to 4$, $III \to 3$, $IV \to 1$; $I \to 4$, $II \to 2$, $III \to 3$, $IV \to 1$; *min* cost = Rs. 59].

18. Four engineers are available to design four projects. Engineer 2 is not competent to design the project B. Given the following time estimates needed by each engineer to design a given project, find how should the engineers be assigned to projects so as to minimize the total design time of four projects.

		Projects.			
		A	B	C	D
Engineer	1	12	10	10	8
	2	14	Not Suitable	15	11
	3	6	10	16	4
	4	8	10	9	7

[Ans. $1 \to B$, $2 \to D$, $3 \to A$, $4 \to C$; total time = 36 hours.]

19. The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows :

	A	B	C	D	E
1	62	78	50	101	82
2	71	84	61	73	59
3	87	92	111	71	81
4	48	64	87	77	80

Find by using the assignment method, the assignment of machinists to jobs that will result in a maximum profit. Which job should be declined ? **[JNTU (B. Tech.) 2003]**

[Ans. $1 \to D$, $2 \to B$, $3 \to C$, $4 \to E$; max. profit = Rs . 376 .]

20. A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are estimated as follows :

Machine

Job	A	B	C	D
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

Solve the problem to maximize the total profit.

[Ans. $2 \to A$, $3 \to B$, $4 \to D$, $6 \to C$, max. profit = 28 .]

21. Find the optimal assingment for the given assignment.

Machine

Job	1	2	3
1	5	7	9
2	14	10	12
3	15	13	16

[I.A.S (Main.) 99]

22. Solve the following assignment problem to maximize production. The data given in the table refers to production in certain units :

Machine

Operator	A	B	C	D
1	10	5	7	8
2	11	4	9	10
3	8	4	9	7
4	7	5	6	4
5	8	9	7	5

[Delhi (MCI) 2000]

23. In a textile sales emporium, four salesman *A, B, C* and *D* are available to 4 counters *W, X, Y,* and *Z*. Each salesman can handle any counter. The service (in hrs.) of each counter when manned by each salesman is given below :

Salesman

Counter	A	B	C	D
W	41	72	39	52
X	22	29	49	65
Y	27	39	60	51
Z	45	50	48	52

How should the salesman be allocated to appropriate counters so as to minimize the service time ?

[IPM (PGDBM) 2000]

24. (a) The solution to assignment problem is inherently degenerate. Explain. **[VTU 2003]**

(b) An Air Transport Co. picks up and delivers freight where customers require. The Co. has two types of air craft *X* and *Y* with equal loading capacities but different operating costs as shown :

Type of Air craft	Cost per KM in Rs.	
	Empty	loaded
X	1.00	2.00
Y	2.00	3.00

The present four locations of the air craft which the Co. is having are :

$$J \to X, K \to Y, L \to Y \text{ and } M \to X$$

Four customers of the Co. located at *A, B, C* and *D* want to transport nearly the same load to their final destinations. The final destinations are at a distance of 600, 300, 100 and 500 KMs from loading points. *A, B, C* and *D* respectively. Distance between locations of air craft and loading points are as follows :

	A	B	C	D
J	300	200	400	100
K	300	100	300	300
L	400	100	100	400
M	200	200	300	200

Determine the optimum allocation and total cost. **[VTU 2003]**

OBJECTIVE QUESTIONS

1. An assignment problem is considered as a particular case of a transportation problem, because
   (a) the number of rows equals the number of columns. (b) all $x_{ij} = 0$ or 1.
   (c) all rim conditions are 1. (d) all of the above.
2. An optimal assignment requires that the maximum number of lines which can be drawn through squares with zero opportunity cost be equal to the number of
   (a) rows or columns. (b) rows and columns. (c) rows + column–1. (d) none of the above.
3. While solving assignment problem. an activity is assigned to a resource through a square with zero opportunity cost because the objective is to
   (a) minimize total cost of assignment. (b) reduce the cost of assignment to zero.
   (c) reduce the cost of that particular assignment to zero. (d) all of the above.
4. The method used for solving an assignment problem is called
   (a) reduced matrix method. (b) MODI method. (c) Hungarian method. (d) none of the above.
5. The purpose of a dummy row or column in an assignment problem is to
   (a) obtain balance between total activities and total resources.
   (b) prevent a solution from becoming degenerate.
   (c) provide the means of representing a dummy problem.
   (d) none of the above.
6. Maximization assignment problem is tansformed into a minimization problem by
   (a) adding each entry in a column from the maximum value in that column.
   (b) subtracting each entry in a column from the maximum value in that column.
   (c) subtracting each entry in the table from the maximum value in that table.
   (d) any one of the above.
7. If there were n workers and n jobs, there would be
   (a) n ! solutions. (b) $(n - 1)$! solutions. (c) $(n\,!)^n$ solutions. (d) n solutions.
8. An assignment problem can be solved by
   (a) simplex method. (b) transportation method. (c) both (a) and (b). (d) none of the above.
9. For a salesman who has to visit n cities. following are the ways of his tour plan
   (a) n !. (b) $(n + 1)$!. (c) $(n - 1)$!. (d) n.
10. The assignment problem
    (a) requires that only one activity be assigned to each resource.
    (b) is a special case of transportation problem.
    (c) can be used to maximize resources.
    (d) all of the above.

Answers

1. (d) **2.** (a) **3.** (a) **4.** (c) **5.** (a) **6.** (c) **7.** (a) **8.** (c) **9.** (c) **10.** (d).

***

13

MULTI-CRITERIA DECISION PROBLEMS
(GOAL PROGRAMMING)

13.1. INTRODUCTION

The purpose of this chapter is to describe the subject called *Goal Programming* or GP and to distinguish the models, methods and applications as a unique subject of study. In addition, the relationship of GP within the fields of *management science/operations research* and multiple criteria decision making will be discussed. Decision making within an organization is often characterized by an attempt to satisfy a set of potentially conflicting objectives as completely as possible in an environment composed of limited resources, divergent interests and an annyoing priorities in order to deal with situations in which all objectives cannot be completely and/or simultaneously satisfied.

13.2. CONCEPT OF GOAL PROGRAMMING

The basic idea of GP has been traced by *Romero* (1992) to a study by *Charnes, Cooper* and *Ferguson* (1955) on executive compensation. According to *Romero* (1992), it was not until *Charnes* and *Cooper's* 1961 linear programming text book. Indeed GP was not even cited as a term in the index of the *Charnes* and *Cooper* (1961) book. Interestingly, it was not presented as a unique or revolutionary methodology, but as an extension of linear programming (LP).

Basically, the method of goal programming consists of formulating an objective function in which optimization comes as close as possible to the specified goals. *Ijiri* (1965) developed the concept of priority factors, assigning different priority levels to goals and different weights for the goals at the same priority level. Lee (1972) and Ignizio (1976) have discussed the subject of goal programming which is an extension of linear programming.

In earlier days, profit maximization was used to be considered as the prime goal of management. However, in today's dynamic business environment, profit maximization is not the only goal management pursues. There are now other priority goals as well. These include social responsibility, public relations, industrial and labour relations etc. Such goals are sought because of outside pressure or voluntary management decisions.

Thus non-economic goals exist and are gaining significance. As such, there is no single universal goal for today's organizations. Management has multiple conflicting objectives to achieve in the context of present business scenario.

The decision criteria for today's managers is multi-dimensional and decision-process involves multiple goals.

Goal Programming as a powerful tool for handling multiple criteria is now being widely used. In this, various goals are expressed in different units of measurement such as rupees, hours, tonnes etc. Manytimes the multiple goals are in conflict and one can be achieved only at the expense of other. Therefore, goals are arranged in order of importance and their contribution to organization's well-being. Also all goal constraints are in linear relationships. With such set-up, the problem can be solved by goal programming.

In goal programming, there is no single objective function as in linear programming. The deviations between the goals and what can be achieved within the given set of constraints are minimized.

The objective function primarily contains deviational variables that represented in two dimensions in the objective functions, a positive and a negative deviation from each sub-goal and for constraint.

The objective function becomes the minimization of these deviations, based on the relative importance or priority assigned to them.

Q. 1. Define Goal Programming. **[IGNOU 99 (Dec.)]**

2. Illustrate multicriteria optimization. **[AIMS (MBA) 2002]**

13.3. GP AS AN EXTENSION OF LP : FORMULATION

Since the origin of GP can be traced to LP, a starting point of GP model, a generally accepted statement of GP model was presented by *Charnes* and *Cooper* (1977).

The formulation of GP model is similar to that of LP model. To do so, all specified goals are ranked in order of their priority. The important advantage of goal programming is that it can be solved by modified version of simplex method.

The general GP model can be stated as follows :

$$\textit{Minimize}: \quad z = \sum_{i=1}^{m} w_i (d_i^- + d_i^+),$$

$$\textit{Subject to}: \quad \sum_{j=1}^{n} a_{ij} x_j + d_i^- - d_i^+ = b_i, \; i = 1, 2, \ldots, m$$

$$\textit{and} \quad x_j, d_i^-, d_i^+ \geq 0, \text{ for all } i, j$$

where m goals are expressed by an m-component column b_i, a_{ij} represents the coefficient for the jth decision variable in the ith constraint, x_j represents a decision variable, w_i represents the weights of each goal, and d_i^- and d_i^+ are deviational variables representing the amount of under-achievement and over-achievement of ith goal respectively.

In case, goals are classified in k ranks, the preemptive priority factors (P_1, P_2, ... and so on) should be assigned to deviational variables d_i^- and d_i^+ according to their order of importance. Here P's are not given actual values, but this is simply a convenient way of indicating that one goal is more important than another. The priority factors have the relationship of $P_j >>>> n P_{j+1}$ ($j = 1, 2, \ldots, k$) where n is very large. This indicates that multiplication by n, however large it may be, cannot make $n P_{j+1}$ greater than P_j. Therefore, lower-priority goal can never be achieved at the expense of higher-priority goal.

At the same priority level, the deviational variables may be given differential weights in the objective function so that deviational variables within the same priority have different cardinal weights.

Since both under and over-achievement of a goal cannot be achieved simultaneously, (either one or both of these deviational variables will be equal to zero, *i.e.* $d_i^- \times d_i^+ = 0$). In other words, it either assumes a positive value or the other must be zero, and vice-versa.

The decision maker must analyse each one of the m goals in terms of whether under or over-achievement of the goal is satisfactory. If over-achievement is acceptable, d_i^+ (called surplus variable in LP) can be removed from the objective function. On the other hand, if under-achievement is acceptable, d_i^- (called the slack variable in LP) can be removed from the objective function. If exact achievement of the goal is derived, both d_i^- and d_i^+ must be included in the objective function, and ranked according to their preemptive priority factors from the ***most important to the least important***. In this manner, *the lower order goals are considered only after the higher goals are achieved.*

Q. What is the difference between linear programming and goal programming ? **[IGNOU 2000 (June)]**

Now we shall discuss the goal model extensions and their solutions more systematically as follows :

13.3-1. Single-Goal Models

To obtain a clear understanding of the GP–LP relationship, we begin with a single-goal problem.

Example 1. *Consider the example where we produce two products. Each product requires time in two production departments : product 1 requires 20 hrs. in department 1 and 10 hrs. in department 2. Product 2 requires 10 hrs in department 1 and 10 hrs. in department 2. Production time is limited in department 1 to 60 hrs. and in department 2 to 40 hrs. Contribution to profits for the two products is Rs. 40 and Rs. 80 respectively. Management's objective (goal) is to maximize profits.*

Solution. The LP formulation of the problem is :

$$\textit{Maximize}: \quad z = 40x_1 + 80x_2 \qquad \ldots(13.1)$$

$$\textit{Subject to}: \quad 20x_1 + 10x_2 \le 60 \qquad \ldots(13.2)$$

$$10x_1 + 10x_2 \le 40 \qquad \ldots(13.3)$$

$$x_1, x_2 \ge 0,$$

where x_1 = number of units of product 1 to produce, and x_2 = number of units of product 2 to produce. The optimal solution to the problem, which can be solved by simplex method, is $x_1 = 0, x_2 = 4$, and z = Rs. 320. Twenty hours of slack time remain in department 1 ($s_1 = 20$), and no time remains in department 2 ($s_2 = 0$).

The goal programming formulation of the problem is :

$$\textit{Minimize}: \quad z = d^- \qquad \ldots(13.4)$$

$$\textit{Subject to}: \quad 20x_1 + 10x_2 \le 60 \qquad \ldots(13.5)$$

$$10x_1 + 10x_2 \le 40 \qquad \ldots(13.6)$$

$$40x_1 + 80x_2 + d^- - d^+ = 1000 \qquad \ldots(13.7)$$

$$x_1, x_2, d^-, d^+ \ge 0,$$

where the x's represent decision variables and the d's represent deviation variables.

The key difference between the LP model [(13.1) through (13.3)] and the GP model [(13.4) through (13.7)] is the objective function, where deviational variables have been used and equation (13.7) has been added. Equation (13.7) appears very much like a constraint, but is actually the goal equation for the model the profit goal in this case. Since we are seeking to maximize profits, we set an arbitrarily high goal, Rs. 1000 in this example. The two non-negative variables, d^- and d^+, are the *deviational variables* for the goal. They represent the amount by which we under-achieve (d^-) or over-achieve (d^+) the profit goal of Rs. 1000, respectively.

While in the majority of applications, both d^+ and d^- deviationals will appear in a goal equation, at most only one of the two variables will take on a positive value in any solution. For example, it is impossible to under-achieve the profit goal of Rs. 1000 at the same time as it is over-achieved. If the goal is achieved *exactly*, both deviational variables will be zero if the goal cannot be achieved then one or the other of the variables will be zero.

Since our objective is to maximize profit, the objective function of the GP model, equation (13.4), contains only a deviational variable (the profit function that appeared in the LP model is written as a goal constraint). Only the deviational variables associated with the goal (objective) appear in the objective function. For this problem only the d^- deviational variable is included. This results from the fact that our objective, in GP form, is to minimize the under-achievement of the profit goal, and since under-achievement is undesirable, we would like to drive d^- as close to zero as possible. If over-achievement of the profit goal were considered undesirable (which is unlikely here since we are considering profits, but likely if we were considering over-time), then only d^+ would be included in the objective function. If the decision maker were interested in achieving the profit goal *exactly*, then both deviational variables would be included in the objective function.

Since the deviational variables can be treated like any other variable, we can solve the GP formulation of the problem with the traditional LP solution algorithm (simplex method), and thereby obtain the final table is shown in Table 13.2. We see that the optimal solution is : $x_1 = 0, x_2 = 4, s_1 = 20, s_2 = 0$, and $d^- = 680$. This solution is identical to the solution for the LP formulation, except for the difference in the z value; for the LP formulation, z = Rs. 320; in the GP formulation z = Rs 680. (The z value is actually – Rs. 680 in Table 13.1, but remember that we are solving a minimization problem using the maximization logic of the simplex method.) The z value for the GP solution reflects the extent by which we under-achieved our profit goal of Rs. 1000, so the actual maximum profit for the problem is Rs. 320 (that is 1000– 680), identical to the LP solution.